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# Stable Manifold Method Unified with Symplectic Algorithm for Nonlinear Optimal Controls

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**Abstract:** This paper proposes an extended stable manifold method that is unified with the Störmer-Verlet method. The stable manifold method is an iterative numerical solver of Hamilton-Jacobi equations in nonlinear optimal control problems. The Störmer-Verlet method is a simple symplectic algorithm that can be applied to Hamiltonian systems. In this extension, the Störmer-Verlet method is used for improving a numerical integration of equivalent Hamiltonian systems of the Hamilton-Jacobi equations in the stable manifold method. The extended stable manifold method is applied to a nonlinear $H^\infty$ control problem to demonstrate its effectiveness.

**Keywords:** Nonlinear optimal controls, symplectic numerical algorithm, nonlinear $H^\infty$ controls

## 1. INTRODUCTION

The stable manifold method is an exact numerical solver of Hamilton-Jacobi (HJ) equations in nonlinear optimal controls [1]. This method offered a feasible approach to solving HJ equations that were considered difficult to solve. The method has been applied to various physical systems [2]. The iterative scheme of the method evaluates the numerical integral of an equivalent Hamilton system of the HJ equation.

The Störmer-Verlet method [3] is a numerical integrator that preserves geometric properties of flows of differential equations, i.e., reversibility, symplecticity, volume preservation, and conservation of first integrals. The structure preservation leads to improved long-time behavior such as the turnpike behavior [4]. The Störmer-Verlet method has been applied to the equivalent Hamiltonian systems of the HJ equations without using the stable manifold method in [5].

This paper proposes the unification of the stable manifold method with the Störmer-Verlet method in contrast to the approach [5]. The numerical integration of the stable manifold method is carried out along a trajectory from the neighborhood of the origin of the system to the outside region. The numerical accuracy decreases in the outside region where nonlinear effects dominate. The unified method modifies the update rule using the Störmer-Verlet method in the outside region.

This paper is organized as follows: First, the basic algorithms of the stable manifold method for nonlinear $H^\infty$ control problems is explained in the successive section. The application of the stable manifold method for the nonlinear $H^\infty$ control problem [6] is introduced as a complex model with numerical difficulty. Next, the extended stable manifold method is explained in terms of the Störmer-Verlet method. Finally, the application of the nonlinear $H^\infty$ control problem for an equivalent two-wheeled vehicle model is presented to show the effectiveness of the proposed method.

## 2. PRELIMINARIES

### 2.1. Nonlinear $H^\infty$ control problems

Let $\mathcal{X}$ be a smooth $n$-dimensional manifold such that $\mathcal{X} \subseteq \mathbb{R}^n$. Let us consider the following control system defined on $\mathcal{X}$:

$$\begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u, \\ y = x, \\ z = h(x) + k(x)u, \end{cases} \tag{1}$$

where $x(t) \in \mathcal{X}$ is the vector of state variables, $u(t) \in \mathcal{U} \subseteq \mathbb{R}^p$ is the vector of control inputs, $w(t) \in \mathcal{W}$ is the vector of disturbance, $y(t) \in \mathbb{R}^n$ is the vector of measured outputs, $z(t) \in \mathbb{R}^q$ is the vector of reference outputs, and $\mathcal{U}$ and $\mathcal{W}$ denote, respectively, the set of admissible controls and the set of admissible disturbances. Note that an admissible function means that it is defined on some time interval and it is piecewise continuous. In (1), $f\colon \mathcal{X} \to \mathcal{V}(\mathcal{X})$, $g_1\colon \mathcal{X} \to \mathcal{M}^{n\times r}(\mathcal{X})$, $g_2\colon \mathcal{X} \to \mathcal{M}^{n\times p}(\mathcal{X})$, $h_1\colon \mathcal{X} \to \mathbb{R}^s$, and $k\colon \mathcal{X} \to \mathcal{M}^{p\times m}(\mathcal{X})$ are assumed to be real vector-valued $C^\infty$-functions, where $\mathcal{V}$ is the vector space of all smooth vector fields over $\mathcal{X}$, and $\mathcal{M}^{i\times j}(\mathcal{X})$ is the ring of $(i \times j)$ matrices over $\mathcal{X}$.

The nonlinear $H^\infty$ control problem can be defined as follows: Let $\gamma > 0$ be a constant that is a design parameter with respect to disturbances. Then, find a control input $u$ satisfying $\|\varSigma\|_{\mathcal{H}^\infty} \leq \gamma$ for the system (1), where $\|\varSigma\|_{\mathcal{H}^\infty}$ is the $H^\infty$ norm defined by

$$\|\varSigma\|_{\mathcal{H}^\infty} = \sup_{w\in\mathcal{L}^2\cap\mathcal{L}^\infty_c\setminus\{0\}} \frac{\|z\|_2}{\|w\|_2}, \quad x(t_0) = 0, \tag{2}$$

for $w \in \mathcal{L}^2 \cap \mathcal{L}^\infty_c \setminus \{0\}$ meaning that $w \in \mathcal{L}^2$ satisfies $\sup_t |w(t)| \leq c$ for some constant $c$ and $w \neq 0$.

It is known that the solution of the nonlinear $H^\infty$ control problem can be given as a solution the Hamilton-Jacobi-Isaac equation (see [6]):

$$\begin{aligned} H(x,p,u,w) &= p^\top f(x) + \frac{1}{4\gamma^2} p^\top g_1(x) g_1^\top(x) p \\ &- \frac{1}{4} \varXi^\top(x,p) K^{-1}(x) \varXi(x,p) + h^\top(x) h(x) = 0, \end{aligned} \tag{3}$$

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† Hana Okada is the presenter of this paper.

where $\Xi(x,p) = g_2^\top(x)p + 2k^\top(x)h(x)$.

This study assumes that

i) $x = 0$ is a unique equilibrium point of the system (1) when $u = 0$ and $w = 0$.

ii) $f(0) = 0$, $h(0) = 0$, and $K(x) = k^\top(x)k(x) > 0$ hold.

iii) There exists a unique solution $x(t, t_0, x_0, u)$ on the time interval $[t_0, \infty] \in \mathbb{R}$ that continuously depends on the initial condition $x_0$.

iv) $k^\top(x)h(x) = 0$ for all $x \in \mathcal{X}$.

### 2.2. Stable manifold method

The stable manifold method [1] calculates a stable manifold of stabilizing solutions of the HJ equation by using the following iterative numerical scheme:

i) Transform the equivalent Hamiltonian system of the HJ equation as

$$\begin{bmatrix} \dot{x}' \\ \dot{p}' \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & -F^T \end{bmatrix} \begin{bmatrix} x' \\ p' \end{bmatrix} + \begin{bmatrix} n_s(t,x',p') \\ n_u(t,x',p') \end{bmatrix} \tag{4}$$

by the coordinate transformation

$$\begin{bmatrix} x' \\ p' \end{bmatrix} = \begin{bmatrix} I & S \\ P & PS+I \end{bmatrix}^{-1} \begin{bmatrix} x \\ p \end{bmatrix}, \tag{5}$$

where we have defined $F = A - RP$ and the higher order terms $n_s$ and $n_u$, $A$ is the first order part of $f(x) = Ax + \mathcal{O}(x^2)$, $P$ and $S$ are the stabilizing solution of the Riccati equation and the solution of Lyapunov equation, respectively:

$$PA + A^\top P - PRP + Q = 0, \tag{6}$$

$$FS + SF^T = F, \tag{7}$$

stabilizing solution means that $F$ is a stable matrix, and the constant matrixes $R$ and $Q$ are derived from the relations $\bar{R}(x) = R_2(x) - (1/\gamma^2)g_1(x)g_1^\top(x) = R + \mathcal{O}(x)$ with $R_2(x) = g_2(x)K^{-1}(x)g_2^\top(x)$ and $\bar{Q}(x) = h^\top(x)h(x) = (1/2)x^\top Qx + \mathcal{O}(x^3)$, respectively.

ii) Calculate sequences $\{x'_k(t,\xi)\}$ and $\{p'_k(t,\xi)\}$ determined by

$$\begin{aligned} & x'_{k+1}(t,\xi) \\ & \quad = e^{Ft}\xi + \int_0^t e^{F(t-s)} n_s(s, x'_k(s,\xi), p'_k(s,\xi)) ds, \\ & p'_{k+1}(t,\xi) \\ & \quad = -\int_t^\infty e^{-F^T(t-s)} n_u(s, x'_k(s,\xi), p'_k(s,\xi)) ds \end{aligned} \tag{8}$$

for a certain parameter $\xi \in \mathbb{R}^n$, where $x'_0(t,\xi) = e^{Ft}\xi$, and $p'_0(t,\xi) = 0$.

iii) By iteratively applying (8), extend a solution along an initial vector $\xi$ in a plain surface spanned by $P$ under the condition that the Hamiltonian of the right side of (3) is sufficiently close to zero.

iv) If a solution passes through a desired initial state of control systems, then the iteration is finished. If not, back to the procedure 2 and try with other $\xi$.

## 3. MAIN RESULT

### 3.1. Stable manifold method unified with Störmer-Verlet scheme

For the initial time $t_0 = 0$, let $t_n = nh$ and $t_{t+1/2} = (n + 1/2)h$ for an integer number $n$ and a time step $h$. Let $p_n = p(t_n)$, $p_{n+1/2} = p(t_{n+1/2})$, $x_n = x(t_n)$, and $x_{n+1/2} = x(t_{n+1/2})$. Then, the iterative scheme of the Störmer-Verlet method [3] is given as follows:

$$\begin{aligned} p_{n+1/2} &= p_n - \frac{h}{2} H_x(x_n, p_{n+1/2}), \\ x_{n+1} &= x_n + \frac{h}{2} \left[ H_p(x_n, p_{n+1/2}) + H_p(x_{n+1}, p_{n+1/2}) \right], \\ p_{n+1} &= p_{n+1/2} - \frac{h}{2} H_x(x_{n+1}, p_{n+1/2}) \end{aligned} \tag{9}$$

or

$$\begin{aligned} x_{n+1/2} &= x_n - \frac{h}{2} H_p(x_{n+1/2}, p_n), \\ p_{n+1} &= p_n + \frac{h}{2} \left[ H_x(x_{n+1/2}, p_n) + H_x(x_{n+1/2}, p_{n+1}) \right], \\ x_{n+1} &= x_{n+1/2} - \frac{h}{2} H_p(x_{n+1/2}, p_{n+1}), \end{aligned} \tag{10}$$

where we have denoted $H_p(x,p) = \partial H/\partial p$, and $H_x(x,p) = \partial H/\partial x$. (9) and (10) include implicit equations, i.e., the parameters of the left-sides depend on the descriptions of the right-side; therefore the left-sides cannot be determined from the right-sides. For example, the first two equations in (9) are implicit, and they should be solved by, e.g., Newton's method. The initial guess of $(x_{n+1}, p_{n+1/2})$ can be given by $(x_n, p_n)$ for an appropriate small $h$. Note that $H_p(x,p)$ and $H_x(x,p)$ in (9) and (10) can be calculated from the equivalent Hamiltonian system of (3):

$$\begin{aligned} \dot{x} &= \frac{\partial H^\top}{\partial p} = f(x) - \frac{1}{2}\left( R_2(x) - \frac{1}{\gamma^2} g_1(x) g_1^\top(x) \right) p, \\ \dot{p} &= -\frac{\partial H^\top}{\partial x} = -\frac{\partial f}{\partial x}^\top(x) p - \frac{1}{2\gamma^2} p^\top \frac{\partial g_1}{\partial x}(x) g_1^\top(x) p \\ &\quad + \frac{1}{4}\left( \frac{\partial}{\partial x} p^\top R_2(x) p \right)^\top - 2\frac{\partial h^\top}{\partial x}(x) h(x). \end{aligned} \tag{11}$$

The proposed method is defined as follows:

1. In the first numerical scheme of the stable manifold method, calculate the following two update rules in (4), and (9) (or (10)).
2. Calculate the time variations of the Hamiltonian (3) by substituting $(\dot{x}', \dot{p}')$ in (4), and $(x_{n+1}, p_{n+1})$ in (9) (or (10)) into $(x, p)$ of (3), respectively.
3. Determine an update rule consisting of the sum of the two rules with weights: $\alpha$ for $(\dot{x}', \dot{p}')$ and $1 - \alpha$ for $(x_{n+1}, p_{n+1})$ according to the variations of the Hamiltonian.

On the other hand, in the approach [5], (9) (or (10)) is independently used without iterative scheme (8).

### 3.2. Controlled vehicle model

Then, the equivalent two-wheeled vehicle model with respect to yawing without rolling and pitching motions is given as follows:

$$f(x) = \begin{bmatrix} -\dfrac{\sin\beta}{mV_0}F_x + \dfrac{\cos\beta}{mV_0}F_y - r \\ \dfrac{2l_f}{I}C_f\cos\delta - \dfrac{2l_r}{I}C_r \\ r \\ 0 \\ V_0\sin(\beta+\theta) \end{bmatrix}, \tag{12}$$

$$g_1(x) = \begin{bmatrix} \cos(\beta+\theta)/(mV_0) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^\top, \tag{13}$$

$$g_2(x) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^\top, \tag{14}$$

where the control input $u$ is the steering angle speed, and

$$x = \begin{bmatrix} \beta & r & \theta & \delta & Y \end{bmatrix}^\top, \quad w = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^\top$$

consisting of the slip angle $\beta$ at center of gravity (COG) [rad], the yaw rate $r$ [rad/s], the direction $\theta$ [rad], the steering angle $\delta$ [rad], and the lateral position $Y$ of the vehicle (the vertical position is omitted), and $w_1$ and $w_2$ are disturbances. The translational forces $F_x$ and $F_y$, and the cornering force of each wheel $Y_i$ are defined as follows:

$$F_x = 2Y_f\sin(\beta_f+\delta) + 2Y_r\sin\beta_r, \tag{15}$$

$$F_y = 2Y_f\cos(\beta_f+\delta) + 2Y_r\cos\beta_r, \tag{16}$$

$$Y_i = C_i\cos\beta_i \tag{17}$$

for $i = \{f, r\}$ that means the front and the rear wheels, respectively, where $\beta_i$ is the slip angle of wheels [rad], $C_i$ is the lateral force of wheels [N], $C_i$ and $\beta$ are related by

$$C_i = \mu N_i\sin[a\tan^{-1}\{b\beta_i - c(b\beta_i - \tan^{-1}(b\beta_i))\}], \tag{18}$$

where $a = 1.23$, $b = 3.25$ and $c = -6.00$ are experimental parameters, $\mu = 0.2$ is a friction constant between road surface and tire, and $N_f = 5.48$ and $N_r = 4.21$ are vertical loads of each wheel.

In the controlled model, the following physical parameters are used: the constant speed $V_0 = 17.7$ [m/s], the mass $m = 990$ [kg], the moment of inertia $I = 683$ [kg·m$^2$], the distance from front axle to COG $l_f = 1.0$ [m], and the distance from rear axle to COG $l_r = 1.3$ [m],

### 3.3. Simulation results

In the numerical experiments, we applied the weight ratios: $\alpha = \{0.0, 0.02, 0.05, 0.1\}$ to the update vectors calculated by the Störmer-Verlet method. The shooting trajectories of the extended stable manifold method for the nonlinear $H^\infty$ problem of the controlled model (14) are shown in Fig. 1. The trajectory obtained via the original stable manifold method is depicted by the bold solid line in Fig. 1. Despite their resemblance, the trajectories of the proposed method exhibit progressive deformation.

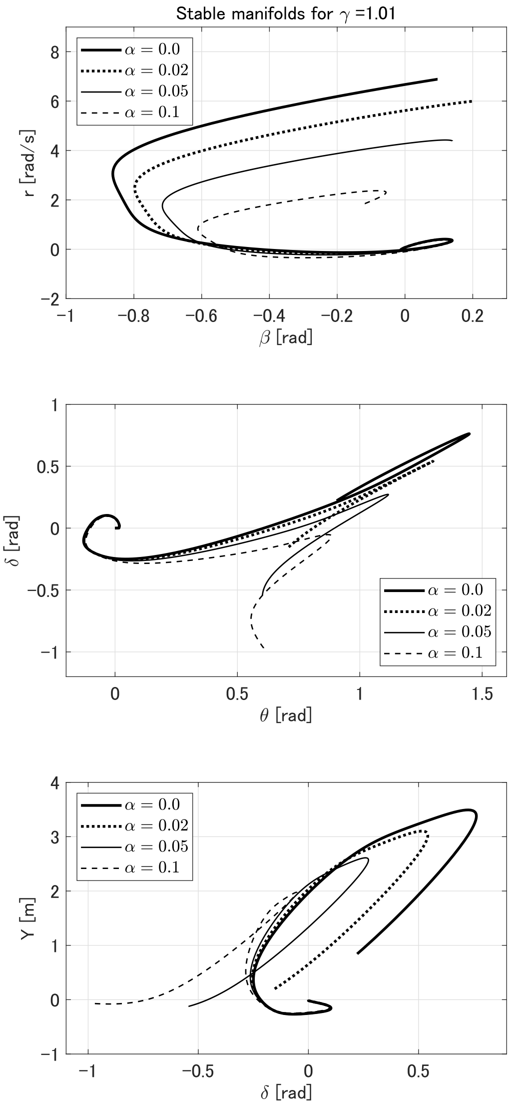

Fig. 1 Stable manifold methods

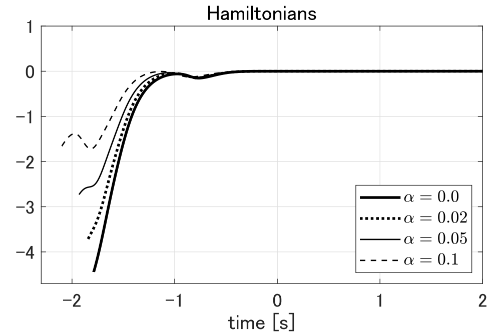

Fig. 2 Hamiltonians

The trajectories are computed from a common initial condition located near the origin along the reverse time axis. Therefore, the trajectories in the actual time approach the origin from outside its neighborhood. The reason why the starting points of the trajectories differ is that they are computed from the origin by tracing backward in time, and a small perturbation at the initial computation time is amplified due to the system's nonlinearity. The trajectories constitute a subset of the entire stable manifold that includes all trajectories of the control problem.

The values of the corresponding Hamiltonians are shown in Fig. 2. The time axis represents the computational time, which proceeds in reverse. Therefore, the terminal points in the negative time domain correspond to the initial states of the trajectories. As the distance from the origin increases, the Hamiltonians become larger and the accuracy deteriorates. The figure shows that the proposed method decreases the Hamiltonian compared to the original method.

## 4. CONCLUSION

In this paper, the extended stable manifold method unified with the Störmer-Verlet method is presented. Compared to the extended method, the original stable manifold method can compute solutions more rapidly. However, we have confirmed the possibility of improving convergence rates of the calculations by means of the extended method.

We will further investigate a formal theory regarding the convergence of the proposed method will be established. In particular, attempts in terms of convergence rate for each term included in Hamiltonians might be important for complex controlled systems such as nonlinear $H^\infty$ controlled systems.

## 5. ACKNOWLEDGEMENT

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