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Design of Robust Static Output Feedback Controllers for Uncertain Polynomial Systems: A Two-step Convex Approach

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Abstract: This paper presents a new framework for static output feedback stabilization of uncertain polynomial systems with bounded actuators. The design procedure is divided into two steps. In the first step, parameter-dependent state-feedback controller is obtained to stabilize the given system with a guaranteed bound on the input magnitude. Then, this controller is used as input to the second step to synthesize a robust static output-feedback gain which is independent of the uncertain parameters. In both steps, the synthesis conditions are given in terms of parameter-dependent linear matrix inequalities which are solely convex in the decision variables and hence do not require any iteration. For the both steps, a parameter-dependent Lyapunov function is employed to reduce the conservativeness of the design conditions. The proposed approach leads to enhanced static output feedback design with more computationally tractable formulation than the existing iterative approaches. Effectiveness of the proposed approach is demonstrated by numerical experiments.

Keywords: Static output feedback, Two-steps design, Parameter-dependent LMIs, Input bounds

1. INTRODUCTION

Design of stabilizing controllers for nonlinear systems is known to be a challenging problem in control theory. Due to recent development of a new relaxation technique in polynomial optimization, construction of certain classes of Lyapunov functions, which are polynomial or rational in state variables, as well as corresponding state-feedback controllers for polynomial systems can be performed in efficient and systematic ways. More precisely, joint search of the Lyapunov matrices and the controllers can be transformed into new decision variables with associated convex constraints called parameter-dependent linear matrix inequalities (PDLMIs) [22]. This transformation is based on a change-of -variable technique [2] in the linear matrix inequality (LMI) formulation for controller synthesis of linear systems. The obtained PDLMIs can be approximated into standard LMIs using some recent computational tools such as the sum-of-squares (SOS) technique [3, 16, 22]. Some developments in this field have been reported in [6–8, 18] for polynomial Lyapunov functions, and in [9, 12, 18, 23] for rational Lyapunov functions.

In this paper, we develop a new design condition for static output feedback stabilizing controllers for polynomial systems subject to control input constraints. Inspired by the approaches of [1, 21], the design procedure can be divided into two steps. In the first step, a parameter-dependent state-feedback controller is obtained to stabilized the given system with a guaranteed bound on the input magnitude. Then, this controller is used as input to the second step to design a robust static output-feedback gain which is independent of the uncertain parameters. The key feature of the synthesis condition in the step two is to introduce additional slack variable which separates the system matrix and the Lyapunov matrix. This separation makes the resulting controller depend on the slack variable instead of the Lyapunov matrix. As a result, a Lyapunov matrix depending on uncertain parameters can be introduced to reduce conservatism in the design condition. The synthesis conditions in both steps are given in terms of PDLMIs which are solely convex in the decision variables and thus can be efficiently implemented via the SOS technique.

Comparing to the existing works in [4, 15, 19], the current approach does not impose any structure on the Lyapunov matrix. Moreover, the current approach does not require any iteration and hence less computational complexity than the recent work in [9] can be expected. The usefulness of the proposed approach is illustrated via numerical experiments.

We use standard notation in this paper. The symbols I_n and O denote the identity matrix of dimension n and the zero matrix of proper dimension, respectively. For a real symmetric matrix A, the inequality $A \succ O$ means that A is positive definite. Similarly, $A \prec O$ indicates that A is negative definite. The symbol $\mathrm{He}(D)$ stands for $D + D^{\mathrm{T}}$ for any square matrix D. Finally, in a symmetric block matrix, the symbol "*" represents terms induced by symmetry.

2. UNCERTAIN POLYNOMIAL SYSTEMS

Consider polynomial systems with time-invariant uncertain parameter $\theta \in \mathbb{R}^p$:

$$\begin{aligned} \dot{x} &= A(x,\theta)x + B(x,\theta)u \\ y &= C(x,\theta)x, \end{aligned} \tag{1}$$

† Tanagorn Jennawasin is the presenter of this paper.

where $A(x,\theta)$, $B(x,\theta)$, and $C(x,\theta)$ are polynomial matrices in x and θ. We assume throughout this section that the parameter θ belongs to the set

$$\Theta = \{\theta \in \mathbb{R}^p \mid c_k(\theta) \geq 0,\ k = 1,2,\ldots,l_\Theta\}$$

with polynomials $c_k(\theta)$'s.

The following parameter-dependent Lyapunov function

$$V(x,\theta) = x^{\mathrm{T}} P^{-1}(x,\theta)x,$$

is considered, where $P(x,\theta)$ is a polynomial matrix in (x,θ) and $P(x,\theta) \succ O$.

We define the level set with respect to the Lyapunov function $V(x) = x^{\mathrm{T}} P^{-1}(x,\theta)x$, for each $\theta \in \Theta$ as follows.

$$\mathcal{V}_\theta = \{x \mid x^{\mathrm{T}} P^{-1}(x,\theta)x \leq 1,\ \theta \in \Theta\}.$$

Moreover, the following set is considered as an estimate of the domain of attraction of the closed-loop system:

$$\mathcal{V}_\Theta = \cup_{\theta \in \Theta} \mathcal{V}_\theta.$$

For local stabilization, the static output feedback control law $u = K(y)y$ is required to keep the closed-loop trajectory inside $\mathcal{X}$, and to satisfy the magnitude constraints

$$|u_j| \leq \mu_j,\ \forall x \in \mathcal{V}_\Theta,\ j = 1,2,\ldots,m. \quad (2)$$

3. MAIN RESULTS

We employ in this section a two-stage design procedure [1, 13, 21] for the static output feedback controller. In particular, a parameter-dependent state feedback controller is designed in the first stage. Then the resulting state feedback gain is used to find a stabilizing output feedback controller in the second stage.

A design condition for the state feedback controller can be cast as a set of parameter-dependent linear matrix inequality (PDLMI) constraints, as state in the following theorem:

Theorem 1: Given the region $\mathcal{X}$, the parameter set Θ, the bounds μ_j for $j = 1,2,\ldots,m$. Suppose that there exist a polynomial matrix $F(x,\theta)$ and a symmetric polynomial matrix $P(x,\theta)$ such that

$$P(x,\theta) \succ O,\quad \forall x \in \mathcal{X},\ \forall \theta \in \Theta \quad (3)$$

$$\begin{aligned} &A(x,\theta)P(x,\theta) + P(x,\theta)A^{\mathrm{T}}(x,\theta) + B(x,\theta)F(x,\theta) \\ &\quad + F^{\mathrm{T}}(x,\theta)B^{\mathrm{T}}(x,\theta) - \sum_{i=1}^{n} \frac{\partial P(x,\theta)}{\partial x_i}(A_i(x,\theta)x) \\ &\quad - \sum_{j=1}^{m} \pm(\mu_j \sum_{i=1}^{n} [\frac{\partial P(x,\theta)}{\partial x_i} B_{ij}(x,\theta)]) \prec O, \quad (4) \\ &\qquad \forall x \in \mathcal{X}, \forall \theta \in \Theta \end{aligned}$$

$$\begin{bmatrix} P(x,\theta) & * \\ e_j^{\mathrm{T}} F(x,\theta) & \mu_j^2 \end{bmatrix} \succ O, \quad (5)$$
$$\forall x \in \mathcal{X}, \forall \theta \in \Theta,\ j = 1,\ldots,m,$$

where $A_i(x,\theta)$ and $B_{ij}(x,\theta)$ denote the i-th row of $A(x,\theta)$ and the (i,j)-element of $B(x,\theta)$, respectively. Then, the system (1) is asymptotically stabilized by the state-feedback controller $u = L(x,\theta)x$ with $L(x,\theta) = F(x,\theta)P^{-1}(x,\theta)$. Furthermore, if $\mathcal{V}_\Theta \subset \mathcal{X}$ holds, the region $\mathcal{V}_\Theta$ is a stability region of the closed-loop system and the control input satisfies the magnitude constraints $|u_j| \leq \mu_j,\ j = 1,2,\ldots,m$ whenever $x \in \mathcal{V}_\Theta$. □

Remark 1: The symbol $\sum_{i=1}^{n} \pm(\cdot)$ in (4) means that the inequality must be satisfied for every combination of $+(\cdot)$ and $-(\cdot)$.

The proof can be done along the similar line as in [10], and thus details are omitted. Inequality (5) implies that the closed-loop system satisfies the input magnitude constraints in (2). Positivity of the Lyapunov function $V(x)$ is implied by (3), and negativity of $\dot{V}(x)$ is implied by (4) together with (2).

If the degrees of the polynomial-matrix variables $P(x,\theta)$ and $F(x,\theta)$ are fixed a priori, the constraints (3)–(5) are indeed PDLMIs in coefficients of $P(x,\theta)$ and $F(x,\theta)$. Moreover, if the region $\mathcal{X}$ is characterized by polynomial inequalities, the SOS relaxation methodology in [22] can be applied to search for $P(x,\theta)$ and $F(x,\theta)$.

Note here that the parameter-dependent controller gain $L(x,\theta)$ cannot be constructed in practice because it depends on the unknown parameter θ. However, the controller gain $L(x,\theta)$ obtained from the design in the first stage can be used to search for a static output feedback gain in the second stage. A design condition for the static output feedback gain is summarized in the following theorem:

Theorem 2: Given the region $\mathcal{X}$, the bounds $\mu_j,\ j = 1,2,\ldots,m$, and α, β, and the polynomial matrix $L(x,\theta)$ such that $A(x,\theta)x + B(x,\theta)L(x,\theta)$ is asymptotically stable in the region $\mathcal{X}$ for all $\theta \in \Theta$. If there exist the symmetric polynomial matrix $P(x,\theta)$, and the polynomial matrices $F(x,\theta)$, $G(x,\theta)$, $H(y)$, $J(y)$ such that

$$P(x,\theta) \succ O,\quad \forall x \in \mathcal{X},\ \forall \theta \in \Theta \quad (6)$$

$$\begin{bmatrix} \mathrm{He}(F(x,\theta)A(x,\theta) + F(x,\theta)B(x,\theta)L(x,\theta)) + M(x,\theta) & * & * \\ P(x,\theta) - F(x,\theta)^{\mathrm{T}} + G(x,\theta)A(x,\theta) + G(x,\theta)B(x,\theta)L(x,\theta) & -\mathrm{He}(G(x,\theta)) & * \\ B(x,\theta)^{\mathrm{T}}F(x,\theta)^{\mathrm{T}} + J(y)C(x,\theta) - H(y)L(x,\theta) & B(x,\theta)^{\mathrm{T}}G(x,\theta)^{\mathrm{T}} & \mathrm{He}(H(y)) \end{bmatrix} \prec O, \forall x \in \mathcal{X},\ \forall \theta \in \Theta \quad (7)$$

$$\begin{bmatrix} -1 & e_j^{\mathrm{T}}L(x,\theta) & 1 \\ * & \mu_j^2 P(x,\theta) & (J(y)C(x,\theta) - H(y)L(x,\theta))^{\mathrm{T}} \\ * & * & \mathrm{He}(H(y)) \end{bmatrix} \prec O,$$
$$\forall x \in \mathcal{X},\ \forall \theta \in \Theta,\ j = 1,2,\ldots,m \quad (8)$$

where

$$M(x,\theta) = -\sum_{j=1}^{m} \pm(\mu_j \sum_{i=1}^{n} [\frac{\partial P(x,\theta)}{\partial x_i} B_{ij}(x,\theta)]).$$

Then the static output feedback controller gian $K(y) = H^{-1}(y)J(y)$ stabilizes the system (1). Furthermore, if

$\mathcal{V}_\Theta \subset \mathcal{X}$ holds, the region $\mathcal{V}_\Theta$ is a stability region of the closed-loop system and the control input satisfies the magnitude constraints $|u_j| \le \mu_j,\ j = 1, 2, \ldots, m$ whenever $x \in \mathcal{V}_\Theta$.

□

The constraints (6)-(8) in Theorem 2 are PDLMIs, which depend polynomially on the parameters x and θ. Hence, they can be translated to standard LMIs via the SOS technique. Moreover, the new design condition in Theorem 2 has several advantages comparing with the previous static output feedback design condition proposed in [9]. The advantages can be described into threefold as below:

1. The constraints of the new design condition are solely convex in the design variables, and thus no iterative algorithm is necessary.
2. The previous design condition in [9] involves a transformation matrix $T(x,\theta)$ such that $C(x,\theta)T(x,\theta) = \begin{bmatrix} I & O \end{bmatrix}$. The matrix $T(x,\theta)$ is usually rational function of (x,θ) when the output matrix $C(x,\theta)$ is polynomial in (x,θ), and thus makes the design condition depends rationally on the same parameter. In order to maintain polynomial structure in the design condition, the matrix C is limited to be a constant. The new design condition, however, does not involve such a transformation matrix, and thus can be applied to a larger class of systems where the matrix $C(x,\theta)$ is polynomial in (x,θ).
3. As opposed to [9], no specific structure is imposed on the design variables $H(y)$ and $J(y)$. The new design condition is expected to less conservative than that of [9]

4. A NUMERICAL EXAMPLE

In this section, We provide a numerical example to illustrate the underlined ideas presented in Section 3. All the examples are executed on Matlab 2019, by using YALMIP [11] interface with SDPT3 as an LMI solver.

Example 1: Consider the following uncertain polynomial system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= A(x,\theta)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B(x,\theta)u \\ y &= \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

where

$$A(x,\theta) = \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 - \frac{3}{4}\theta_1 x_2^2 & \frac{1}{4} - x_1^2 - \frac{1}{2}\theta_2^2 x_2^2 \\ 0 & 0 \end{bmatrix}$$

$$B(x,\theta) = \begin{bmatrix} 0 \\ \theta_1^2 + 1.2\theta_2^2 + 1 \end{bmatrix}$$

The uncertain parameter $(\theta_1, \theta_2) \in \mathbb{R}^2$ is assumed to belong to the uncertainty set $\Theta_\gamma = \{(\theta_1,\theta_2) \mid \theta_1^2 + \theta_2^2 \le \gamma^2\}$ where the parameter γ indicates the size of uncertainty set.

In this example, we design an static output feedback controller $u = K(y)y$ for the uncertain system using Theorem 2 and compare the results with those of [9]. The Lyapunov function candiate considered here is $V(x) = x^{\mathrm{T}}P^{-1}(x,\theta)x$, where $P(x,\theta)$ is a polynomial matrix of degree 2 in x. Note here that when applying the two-step design approach in the current paper, a parameter dependent controller gain $L(x,\theta)$ is firstly designed in Thorem 1, where $L(x,\theta)$ is a polynomial matrix of degree 3 in both x and θ.

We firstly find the maximum value of γ such that the closed-loop system is asymptotically stable. The results when we vary input bounds μ and the degree of $P(x,\theta)$ in θ is summarized in Table 1.

Table 1 Comparison of the maximum radius γ of the uncertainty set such that the closed-loop is stabilized.

[9]	Degree of $P(x,\theta)$ in θ		
μ	0	1	2
1	0.5938	0.7066	0.7987
1.5	0.7564	0.8633	0.9594
2	0.9472	1.0459	1.0897
3.5	1.4326	1.6475	1.9332
Theorem 2	Degree of $P(x,\theta)$ in θ		
μ	0	1	2
1	0.7248	0.8891	1.1241
1.5	0.8361	0.9724	1.2305
2	0.9857	1.2220	1.3858
3.5	1.5614	1.6600	1.9527

We can see in Table 1 that Theorem 2 provides larger maximum values of γ in all cases in the table, where the maximum improvement of 40.94 % occurs when $\mu = 1$ and the degree of $P(x,\theta)$ in θ is 2. In other words, the controller obtained from proposed two-step design can stabize the uncertain system in larger uncertainty regions than that obtained from the iterative approach in [9].

5. CONCLUSIONS

A new sufficient condition has been proposed for the static output feedback synthesis of uncertain polynomial systems. The proposed condition employs a two-stage design procedure, where the resuting design conditions are solely convex in the decision variables with no additional structures imposed. The provided numerical examples show that the controllers obtained from the proposed two-stage design approach outperform those obtained from the existing iterative approach. Extension to design of controllers with more complicated structures, like PID or reduced-order dynamic output feedback controllers is under consideration as a possible future research.

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