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Frequency Response Analysis for Homogeneous Systems

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Abstract: In this paper, we reveal a novel characteristic in frequency analysis for homogeneous systems. Firstly, we show that every homogeneous control system with a sinusoidal input has an equivalent homogeneous closed-loop system. Particularly, every solution of a homogeneous system for an arbitrary amplitude sinusoidal input can be calculated from the solution under the sinusoidal input with amplitude one. Moreover, the Bode gain plot is mathematically valid for any homogeneous system. Finally, we confirm the effectiveness of the frequency analysis by computer simulation.

Keywords: nonlinear system, frequency response, homogeneous system

1. INTRODUCTION

Homogeneous systems, one of the canonical systems of nonlinear control systems, have attracted attention in recent decades; the most significant characteristic of homogeneous systems is the continuous finite-time control [1][2]. In actual experiments, local homogeneous control exhibits superior performance [3].

For actual systems, responses to sinusoidal waves are indispensable [5][6]. In particular, homogeneous filters have been attracting attention recently [7][8].

However, responses of homogeneous systems to sinusoidal waves were examined only by numerical computation; mathematical properties have not been discussed.

In this paper, we reveal a novel characteristic in frequency analysis for homogeneous systems. Firstly, we show that every homogeneous control system with a sinusoidal input has an equivalent homogeneous closed-loop system. Particularly, every solution of a homogeneous system for an arbitrary amplitude sinusoidal input can be calculated from the solution under the sinusoidal input with amplitude one. Moreover, we show that the magnitude of state variables is meaningful; surprisingly, the Bode gain plot is mathematically valid for any homogeneous system!

Finally, we confirm the effectiveness of the frequency analysis by computer simulation of a finite-time stable homogeneous system.

2. PRELIMINARIES

This section briefly introduces definitions and basic properties of homogeneous functions. Throughout the paper, we use the following notations. $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$; $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$; $\|\cdot\|$ denotes (standard) Euclidean norm; for a Lebesgue measurable function $d : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ with a positive integer m, $\|d\|_\infty = \operatorname{ess\,sup}_{t\in[0,+\infty)} \|d(t)\|$; function $d \in \mathcal{L}_\infty$ if $\|d\|_\infty < +\infty$; a continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing; a function α is said to belong class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\alpha(s) \to +\infty$ as $s \to +\infty$; a continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong class $\mathcal{KL}$, if for fixed t, $\beta(r, t)$ is class $\mathcal{K}$, and for fixed r, that is decreasing and $\lim_{t\to+\infty} \beta(r, t) = 0$.

While a few frameworks regarding homogeneous systems have been proposed, we consider r-homogeneous systems in this study; the section introduces definitions and basic properties of r-homogeneous systems.

Definition 1 (Dilation). *Consider $x \in \mathbb{R}^n$ and dilation exponent $r = [r_1, \ldots, r_n] \in \mathbb{R}^n_{>0}$. Mapping $\Delta^r_\varepsilon(x)$ defined by the following equation is said to be a homogeneous dilation depending on $\varepsilon > 0$:*

$$\Delta^r_\varepsilon(x) = [\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n]^T. \tag{1}$$

Definition 2 (r-homogeneous function). *Consider dilation exponent $r \in \mathbb{R}^n$. Then, a function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be an r-homogeneous function of degree k if the following equality holds:*

$$V(\Delta^r_\varepsilon(x)) = \varepsilon^k V(x). \tag{2}$$

An essential r-homogeneous function is an r-homogeneous q-norm defined as follows [5].

Definition 3 (r-homogeneous q-norm). *Suppose $q \in [1, +\infty)$. A function $\|\cdot\|_{r,q} : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows is said to be an r-homogeneous q-norm:*

$$\|x\|_{r,q} = \left(\sum_{i=1}^n |x_i|^{\frac{q}{r_i}} \right)^{\frac{1}{q}} \tag{3}$$

By direct implication of the definition of r-homogeneous norm, the following lemma holds.

Lemma 1. *Any r-homogeneous norm is an r-homogeneous function of degree 1 regardless of q.*

Moreover, homogeneous norms are compatible as the following lemma [5, Lemma 1]:

Lemma 2. *Consider two homogeneous norms $\|\cdot\|_{r_1,q_1}$ and $\|\cdot\|_{r_2,q_2}$ with (possibly) different exponents r_1 and r_2, and constants q_1 and q_2. Then, there exist two class $\mathcal{K}_\infty$ functions α_1 and α_2 such that the following inequality holds for all $x \in \mathbb{R}^n$:*

$$\alpha_1(\|x\|_{r_2,q_2}) \leq \|x\|_{r_1,q_1} \leq \alpha_2(\|x\|_{r_2,q_2}). \tag{4}$$

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† Madoka Murakami is the presenter of this paper.

Note that if $r_i = 1$ for all $i = 1, \dots, n$ and $q = 2$, $\|x\|_{r,q} = \|x\|$ (standard Euclidean norm). Hence, inequality (4) implies

$$\alpha_1 \left(\|x\|_{r,q}\right) \le \|x\| \le \alpha_2 \left(\|x\|_{r,q}\right). \tag{5}$$

Definition 4 (r-homogeneous vector field)**.** *Consider dilation exponent $r \in \mathbb{R}^n$. Then, a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be an r-homogeneous vector field of degree τ if the following equality holds:*

$$f(\Delta_\varepsilon^r(x)) = \varepsilon^\tau \Delta_\varepsilon^r(f(x)). \tag{6}$$

Moreover, we say the r-homogeneous differential equation if $f(x)$ in the right-hand side of the following differential equation is homogeneous of degree τ with respect to dilation exponent r:

$$\dot{x} = f(x). \tag{7}$$

The most prominent characteristic of an r-homogeneous differential equation is the convergence rate summarized as the following theorem [2]:

Theorem 1. *Consider r-homogeneous differential equation* (7) *with continuous f. Then, the following properties hold.*

- *If $\tau < 0$, the origin is finite-time stable; that is, for any $x_0 \in \mathbb{R}^n$ and solution $\varphi(0)$ such that $\varphi(0) = x_0$, there exists a constant $T > 0$ depending on the choice of x_0 such that $\varphi(t) = 0$ for all $t > T$.*
- *If $\tau = 0$, the origin is exponentially stable.*
- *If $\tau > 0$, the origin is practically fixed time stable; that is, for any $r > 0$, there exists a constant $T > 0$ depending on r such that $\|\varphi(t)\| < r$ for any solution $\varphi(t)$ all $\varphi(0) \in \mathbb{R}^n$ and $t \ge T$.*

Remark 1. *Note that we cannot guarantee that a solution of differential equation* (15) *is unique due to the assumption that f is only continuous; we employ a notation φ as a solution of differential equation instead of x; however, we use $x(t)$ as the solution of* (15) *if any confusion does not arise.*

Definition 5 (r-homogeneous system)**.** *Consider dilation exponents $r = [r_1, \dots, r_n] \in \mathbb{R}^n_{>0}$ and $s \in \mathbb{R}_{>0}$, and the following control system:*

$$\dot{x} = f(x, u), \tag{8}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^1$, and $f : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}$ is a continuous mapping.

Then (8) is said to be an r-homogeneous control system of degree τ if the following equality holds:

$$f(\Delta_\varepsilon^r(x), \Delta_\varepsilon^s(u)) = \varepsilon^\tau \Delta_\varepsilon^r(f(x, u)). \tag{9}$$

The following proposition is the most crucial property of r-homogeneous systems.

Proposition 1. *Consider r-homogeneous system (8) and solution $\varphi : \mathbb{R} \to \mathbb{R}^n$. Then, mapping $\hat{\varphi} : \mathbb{R} \to \mathbb{R}^n$ is also a solution:*

$$\hat{\varphi}(t) = \Delta_\varepsilon^r(\varphi(\varepsilon^\tau t)). \tag{10}$$

The proof of the proposition is the same as Proposition 1 in [10].

Another important property of r-homogeneous systems is the input-to-state stability, which is defined as follows:

Definition 6 (ISS[11])**.** *Consider a nonlinear system*

$$\dot{x} = f(x, u), \tag{11}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^1$ satisfying $u \in \mathcal{L}_\infty$; $f : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}$ is a continuous mapping. Then, system (11) *is said to be input-to-state stable (ISS) if there exist a class $\mathcal{KL}$ function $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a class $\mathcal{K}$ function $\gamma : \mathbb{R} \to \mathbb{R}$ such that for any bounded input u, the following inequality holds:*

$$\|x(t)\| \le \beta(\|x(0)\|, t) + \gamma(\|u(\cdot)\|_\infty), \quad \forall t \ge 0. \tag{12}$$

For r-homogeneous systems, the following theorem holds [9].

Theorem 2. *Any r-homogeneous system* (8) *with respect to dilation exponent* (r, s) *is ISS.*

3. PROBLEM STATEMENT

In this study, we consider a frequency response problem for homogeneous systems; that is, we analyze the response of (8) under $u = A \sin \omega t$; more precisely, we consider the following system:

$$\dot{x} = f(x, z_1), \tag{13}$$

$$z_1 = A \sin \omega t, \tag{14}$$

where $x, z_1 \in \mathbb{R}$, and (13) is r-homogeneous of degree τ with respect to dilation exponent $r \in \mathbb{R}^n$ and $s \in \mathbb{R}^1$. Moreover, we suppose mapping f is continuous with respect to x and z_1.

We analyze the relation of $x(t)$ to A and ω in (14).

4. FREQUENCY ANALYSIS OF HOMOGENEOUS SYSTEM

This section transforms system (13)–(14) into an autonomous homogeneous differential equation.

Lemma 3. *Consider (13). Then, the following system is r-homogeneous of degree τ with respect to dilation exponent $\tilde{r} = [r, s, s] \in \mathbb{R}^{n+2}$:*

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}), \tag{15}$$

where $\tilde{x} = [x, z_1, z_2]^T$ and $\tilde{f}(\tilde{x})$ is defined as follows.

$$\tilde{f}(\tilde{x}) = \begin{bmatrix} f(x, z_1) \\ g_1(z) \\ g_2(z) \end{bmatrix}, \tag{16}$$

where

$$\begin{aligned} g_1(z) &= \omega(z_1^2 + z_2^2)^{\frac{\tau}{2s}} z_2 \\ g_2(z) &= -\omega(z_1^2 + z_2^2)^{\frac{\tau}{2s}} z_1 \end{aligned} \tag{17}$$

Proof. Note that $f(x, z_1)$ is r-homogeneous according to the assumption, and we show homogeneity of (17).

$$\begin{aligned} g_1(\Delta_\varepsilon^{\tilde{r}}(z)) &= \omega\varepsilon^{\tau+s}(z_1^2+z_2^2)^{\frac{\tau}{2s}}z_2 \\ &= \varepsilon^{\tau+s}g_1(z), \qquad (18) \\ g_2(\Delta_\varepsilon^{\tilde{r}}(z)) &= -\omega\varepsilon^{\tau+s}(z_1^2+z_2^2)^{\frac{\tau}{2s}}z_2 \\ &= \varepsilon^{\tau+s}g_2(z). \qquad (19) \end{aligned}$$

Therefore

$$\tilde{f}(\Delta_\varepsilon^{\tilde{r}}(\tilde{x})) = \varepsilon^\tau \Delta_\varepsilon^{\tilde{r}}(f(\tilde{x})). \qquad (20)$$

□

Note that (17) is equivalent to (14) with $A = 1$ under the initial condition $z(0) = [0, 1]$ due to $(z_1^2 + z_2^2)^{\tau/2s} = 1$ for all τ and s, and surprisingly, any r-homogeneous control system with any sinusoidal input can be identified with the r-homogeneous differential equation of degree τ with respect to dilation exponent $\tilde{r} = [r, s, s]$. As mentioned above, (17) has good properties; however, when $\tau < 0$, the right-hand side of (17) is discontinuous at the origin. For the issue, the following lemma holds:

Lemma 4. *Consider* (15). *If* $z_1^2(0)+z_2^2(0) \neq 0$, *there exists at least an interval* $I \subset \mathbb{R}^1$ *and one classical solution* $\varphi(x) : I \to \mathbb{R}^{n+2}$ *to* (15).

Proof. Consider function $W(z) = z_1^2(t) + z_2^2(t)$. Then, $\dot{W} = 0$ for all $z \neq 0$. Therefore, $W(z(t)) = W(z(0))$ for all $t \geq 0$.

Let $B(t) = (z_1^2(t) + z_2^2(t))^{\tau/2s}$; $B(t) = B(0) = B_0 > 0$ for all $t \geq 0$ according to the above discussion. Then, equation (17) can be written as

$$\begin{aligned} \dot{z}_1 &= \omega B_0 z_2 \\ \dot{z}_2 &= -\omega B_0 z_1. \end{aligned} \qquad (21)$$

Hence, solutions of (21) for initial condition $(z_1(0), z_2(0))$ are obtained as follows:

$$z_1(t) = z_2(0)\sin(\omega B_0 t) + z_1(0)\cos(\omega B_0 t), \qquad (22)$$

$$z_2(t) = z_2(0)\cos(\omega B_0 t) - z_1(0)\sin(\omega B_0 t). \qquad (23)$$

Note that (21) is Lipschitz continuous, and we can confirm that solutions (22)–(23) are uniquely determined.

Since (13) is continuous with respect to x and z_1, there exists an interval $I \subset [0, \infty)$ and solution $\varphi_x(t)$ on $t \in I$. Therefore, $\varphi(t) = [\varphi_x(t), z_1(t), z_2(t)]^T$ is a solution to (17). □

Note that the right-hand side of (17) is discontinuous when $\tau < 0$; however, Lemma 4 guarantees that there exists a classical solution if $z_1(0)^2 + z_2(0)^2 \neq 0$.

Lemma 5. *Consider* r*-homogeneous system* (13) *and* $z_1 = \sin\omega t$, *where* $\omega > 0$ *is a constant. Suppose* $\varphi_x(t)$ *is a classical solution of* (13) *defined on* $I \subset [0, \infty)$. *Then,* $\varphi_x(t)$ *is also a solution of* x *in* (15) *under initial condition* $z_1(0) = 0$ *and* $z_2(0) = 1$.

Proof. Let $z_1(t) = \sin\omega t$ and $z_2(t) = \cos\omega t$, and $\varphi_x(t)$ is a also a solution of x in (15). Moreover, $z_1^2(t)+z_2^2(t) = 1$ for all $t \in \mathbb{R}$; the following equations hold.

$$\dot{z}_1 = \omega\cos\omega t = \omega(z_1^2(t) + z_2^2(t))^{\frac{\tau}{2s}}z_2(t), \qquad (24)$$

$$\dot{z}_2 = -\omega\sin\omega t = -\omega(z_1^2(t) + z_2^2(t))^{\frac{\tau}{2s}}z_1(t). \qquad (25)$$

Therefore, $z_1(t)$ and $z_2(t)$ are solutions to (17). Accordingly, $\varphi(t) = [\varphi_x^T(t), z_1(t), z_2(t)]^T$ is a solution to (15). □

Obviously, the converse of Lemma 5 holds as follows.

Lemma 6. *Consider* (15) *with initial condition* $\tilde{x}_0 = [x_0^T, z_{10}, z_{20}]^T = [0, 0, 1]$, *and suppose* $\varphi(t) = [\varphi_x(t), \varphi_{z1}(t), \varphi_{z2}(t)]^T$ *is a solution to* (15) *with initial condition* $\tilde{x}_0$. *Then,* $\varphi_x(t)$ *is also a solution of* (13) *for input* $z_1 = \sin\omega t$.

According to Lemmas 3-6, the following theorem holds as one of the main results in the paper.

Theorem 3. *Consider* r*-homogeneous system* (13) *and constant* $\omega > 0$. *Let* $\varphi_x(t)$ *be a solution of* (13) *under input* $z_1 = \sin\omega t$ *and initial condition* $\varphi_x(0) = x_0 = 0$. *Then, for any constant* $A > 0$, $\hat{\varphi}_x(t) = \Delta^r_{A^{1/s}}\varphi(A^{\tau/s}t)$ *is also a solution of* (13) *for input* $z_1(t) = A\sin A^{\tau/s}\omega t$ *with initial condition* $\varphi_x(0) = 0$.

Proof. Owing to contnuity of f, there exists a classical solution $\varphi_x(t)$ to (13) exists on $t \in I \subset \mathbb{R}_{\geq 0}$ with input $z_1(t) = \sin\omega t$. Then from Lemmas 4 and 5, $\varphi_x(t)$ is also a solution for x of (15) under initial condition

Consider initial conditions $x(0) = 0$, $z_1(0) = 0$ and $z_2(0) = 1$; solutions of z_1 and z_2 are $z_1(t) = \sin\omega t$ and $z_2(t) = \cos\omega t$, respectively. Let $\varphi(t) = [\varphi_x(t), z_1(t), z_2(t)]^T$, and $\hat{\varphi}(t) = \Delta_\varepsilon^{\tilde{r}}(\varphi(\varepsilon^\tau t))$ is also a solution of (15) by Proposition 1; initial condition $\hat{\varphi}(0)$ is calculated as $\varphi_x(0) = 0$, $z_1(0) = 0$ and $z_2(0) = \varepsilon^s$. For the initial condition, the following equation holds according to (22):

$$z_1(t) = \varepsilon^s\sin(\omega\varepsilon^\tau t). \qquad (26)$$

Hence, $\hat{\varphi}_x(t) = \Delta_\varepsilon^r\varphi_x(\varepsilon^\tau t)$ is a solution to $z_1 = \varepsilon^s\sin(\varepsilon^\tau\omega t)$. Substitute $\varepsilon = A^{1/s}$, and for any constant $A > 0$, $\hat{\varphi}(t) = \Delta^r_{A^{1/s}}\varphi_x(A^{\tau/s}t)$ is also a solution of (16) for input $u = A\sin A^{\tau/s}\omega t$ under initial condition $x(0) = 0$. □

Theorem 3 is crucial; when we consider $z_1 = A\sin\omega t$ for (13), each solution $\hat{\varphi}_{xi}(t)$ can be written as follows for $i = 1, \ldots, n$:

$$\hat{\varphi}_{xi}(t) = A^{\frac{r_i}{s}}\varphi_{xi}(A^{-\frac{\tau}{s}}t), \qquad (27)$$

where $\varphi_x(t) = [\varphi_{x1}(t), \ldots, \varphi_{xi}(t), \ldots, \varphi_{xn}(t)]^T$ is a solution of (13) with input $z_1 = \sin\omega t$. This implies that $\hat{\varphi}_x(t)$ has a similar solution locus to $\varphi_x(t)$ with $z_1 = \sin\omega_0 t$, where $\omega_0 = A^{-\tau/s}\omega$. This property is significant, particularly in the case of finite-time control; every r-homogeneous finite-time control system has a negative homogeneous degree $\tau < 0$ according to Theorem 1. For r-homogeneous finite-time control systems, $A^{-\tau/s} < 1$ when $A \ll 1$. This means that the locus of $\hat{\varphi}_{xi}(t)$ is similar to the response $\varphi_{xi}(t)$ to the lower frequency sinusoidal input $z_1 = \sin(\omega_0 t)$ with small amplitude.

Moreover, Theorem 3 is a natural extension to linearity as the following corollary because any linear control system is r-homogeneous of degree 0 with respect to dilation exponent $r = (1, \ldots, 1)$.

Corollary 1. *Consider the case that system* (13) *is linear and constant* $\omega > 0$*. Let* $x(t)$ *be a solution of* (13) *under input* $z_1 = \sin \omega t$ *and initial condition* $x(0) = 0$*. Then, for any constant* $a > 0$*,* $\tilde{x}(t) = a \cdot x(t)$ *is also a solution of* (16) *for input* $u = a \sin \omega t$ *and initial condition* $x(0) = 0$.

5. FREQUENCY RESPONSE OF HOMOGENEOUS SYSTEMS

In the above discussion, we do not suppose asymptotic stability of (13). In this section, we show frequency response properties for asymptotically stable r-homogeneous systems.

Lemma 7. *Consider* r*-homogeneous system* (13)–(14) *with* $A = 1$ *and* $x(0) = 0$*;* $z_1 = \sin \omega t$*, where* $\omega > 0$ *is an arbitrary constant. Let the origin of* (13) *be asymptotically stable and* $\varphi_x(t; \omega)$ *be an arbitrary solution to* $z_1 = \sin \omega t$*. Then, mapping* $M : \mathbb{R} \to \mathbb{R}$ *defined as follows is well-defined for any* $q > 0$*:*

$$M(\omega) = \sup_{t \geq 0} \|\varphi_x(t; \omega)\|_{r,q}. \tag{28}$$

Proof. According to Theorem 2 and the fact that $u(t) = \sin \omega t$ is a continuous function, $\sup_{t \geq 0} \|\varphi_x(t; \omega)\| \leq \gamma(1)$ for any ω. Hence by Lemma 5, there exists a class $\mathcal{K}$ function α_1 such that for any solution φ_x,

$$\sup_{t \geq 0} \|\varphi_x(t; \omega)\|_{r,q} \leq \sup_{t \geq 0} \alpha_1^{-1}(\|\varphi_x(t; \omega)\|) \leq \alpha_1^{-1}(\gamma(1)). \tag{29}$$

Therefore, $M(\omega)$ is well-defined. □

According to Lemma 7, the following theorem holds as the other main result.

Theorem 4. *Consider* r*-homogeneous system* (13)–(14)*, the initial condition* $x(0) = 0$*, function* M *defined in* (28)*, and the set of all solution of* (13)–(14) *under* $x(0) = 0$ *is denoted as* $\mathcal{S}$*.*

Moreover, we consider mapping $\mathcal{M}(\omega, A)$ *defined as follows.*

$$\mathcal{M}(\omega, A) = \frac{1}{A^{\frac{1}{s}}} \sup_{\varphi_x \in \mathcal{S}} \sup_{t \geq 0} \|\varphi_x(t; \omega, A)\|_{r,q}, \tag{30}$$

where $\varphi_x(t; \omega, A)$ *is a solution of* (13) *to input* $z_1 = A \sin \omega t$ *under* $x(0) = 0$*.*

Then, the following equation holds.

$$\mathcal{M}(\omega, A) = \sup_{\mathcal{S}} M(A^{-\frac{\tau}{s}} \omega). \tag{31}$$

Proof. Note that the following equation holds according to Theorem 3 and $A > 0$:

$$\|\hat{\varphi}_x(t; \omega, A)\|_{r,q} = \left(\sum_{i=1}^{n} |\hat{\varphi}_{xi}(t; \omega, A)|^{\frac{q}{r_i}} \right)^{\frac{1}{q}}. \tag{32}$$

According to Theorem 3,

$$\begin{aligned}
&\left(\sum_{i=1}^{n} |\hat{\varphi}_{xi}(t; \omega, A)|^{\frac{q}{r_i}} \right)^{\frac{1}{q}} \\
&= \left(\sum_{i=1}^{n} |A^{\frac{r_i}{s}} \varphi_{xi}(A^{-\frac{\tau}{s}} t; A^{-\frac{\tau}{s}} \omega)|^{\frac{q}{r_i}} \right)^{\frac{1}{q}} \\
&= A^{\frac{1}{s}} \left(\sum_{i=1}^{n} |\varphi_{xi}(A^{-\frac{\tau}{s}} t; A^{-\frac{\tau}{s}} \omega)|^{\frac{q}{r_i}} \right)^{\frac{1}{q}} \\
&= A^{\frac{1}{s}} \|\varphi_x(A^{-\frac{\tau}{s}} t; A^{-\frac{\tau}{s}} \omega)\|_{r,q}.
\end{aligned} \tag{33}$$

Note that

$$\sup_{t \geq 0} \|\varphi_x(A^{-\frac{\tau}{s}} t; A^{-\frac{\tau}{s}} \omega)\|_{r,q} = \sup_{t \geq 0} \|\varphi_x(t; A^{-\frac{\tau}{s}} \omega)\|_{r,q}, \tag{34}$$

and

$$\begin{aligned}
\mathcal{M}(\omega, A) &= \sup_{\varphi_x \in \mathcal{S}} \sup_{t \geq 0} \|\varphi_x(t; A^{-\frac{\tau}{s}} \omega)\|_{r,q} \\
&= \sup_{\mathcal{S}} M(A^{-\frac{\tau}{s}} \omega)
\end{aligned} \tag{35}$$

□

Theorem 4 implies that relative magnitude $\mathcal{M}$ is simply the frequency shift of $M(\omega)$; this means that, if we have obtained the gain diagram for $M(\omega)$, the diagram for another amplitude of sinusoidal input $\mathcal{M}(\omega, A)$ is easily obtained from the diagram for $M(\omega)$.

Theorem 4 analyzes a frequency response with respect to a homogeneous norm. For each state variable x_i, the following corollary holds.

Corollary 2. *Consider* r*-homogeneous system* (13)–(14)*. Then, the following equality holds.*

$$\|\hat{\varphi}_{xi}(t; \omega, A)\|_\infty = A^{\frac{1}{s}} \sup_{t \geq 0} \|\varphi_{xi}(t; A^{-\frac{\tau}{s}} \omega)\|_\infty. \tag{36}$$

6. EXAMPLE

To illustrate the effectiveness of the frequency analysis of r-homogeneous systems, we show an example in this section. Consider the following finite-time low-pass filter system:

$$\dot{x} = -(x - u)^{\frac{1}{3}}, \tag{37}$$

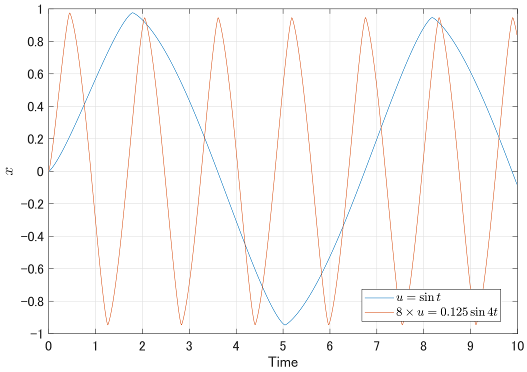

Fig. 1 Comparison of response of x

where x and $u \in \mathbb{R}$.

(37) is r-homogeneous of degree $\tau = -2$ with respect to dilation exponent $r = s = 3$.

Let $x(t)$ be a solution of (37) for input $u = \sin t$ and initial condition $x(0) = 0$. In this example, Theorem 3 implies that $\hat{x}(t) = Ax(A^{-2/3}t)$. For $A = 1/8$, $\hat{x}(t) = x(4t)/8$ is a solution to $u = 0.125 \sin 4t$ (Fig. 1). This means that the solution locus for the low-amplitude and high-frequency signal is similar to that for the high-amplitude and low-frequency signal; that is, a small amplitude input plays a lower-frequency signal in this case.

We show Bode gain plots in Fig. 2 for $A = 1$ and Fig. 3 for $A = 0.01$. Moreover, Bode gain plots for multiple amplitudes and frequencies in Fig. 4. In these figures, vertical axes represent $20 \log_{10} |x(t)|$, and horizontal axes denote ω in log scale. Figure 2 illustrates the frequency response for $u = \sin \omega t$, and Figure 3 one for $u = 0.01 \sin \omega t$. We can confirm that a small magnitude signal plays as a lower magnitude signal for $u = \sin \omega t$.

According to Fig. 2, the magnitude of x for $\omega = 0.01$ is approximately 0[dB]$\simeq$ 1. We illustrate the time response with $u = \sin 0.01t$ in Fig. 5. The blue line shows a time history of x and the red dashed line illustrates $\sin 0.01t$ itself. We can see that the blue line overlaps with the red line; $x(t)$ is almost the same as $\sin 0.01t$. Theorem 3 claims that the input $u = 10^{-6} \sin 100t$ gives a similar performance as $u = \sin 0.01t$. Figure 6 illustrates time histories of $x(t)$ with $u = 10^{-6} \sin 100t$ and the input itself. In this case, we can also confirm that the blue line overlaps with the red one. Figure 7 illustrates time histories of $x(t)$ with a larger magnitude signal $u = \sin 100t$ and the input. We can find that $x(t)$ is more suppressed than the input signal $\sin 100t$. This implies that a r-homogeneous system demonstrates an amplitude-dependent frequency response.

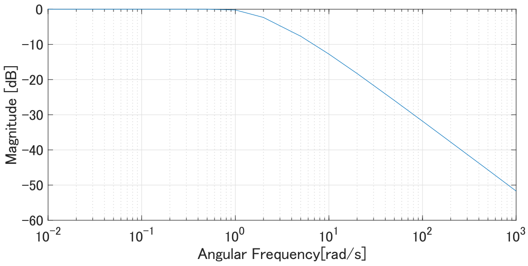

Fig. 2 Frequency Response for $u = \sin \omega t$

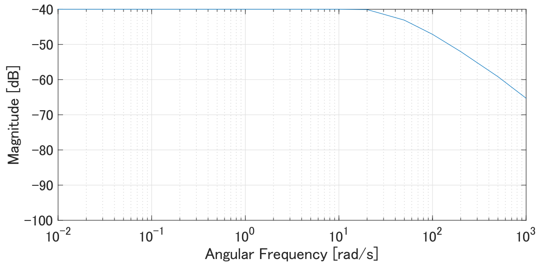

Fig. 3 Frequency Response for $u = 0.01 \sin \omega t$

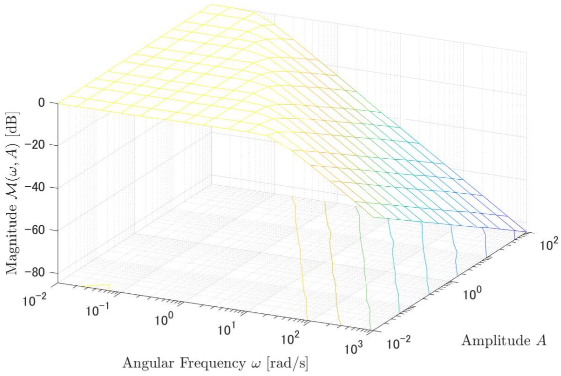

Fig. 4 Bode Gain Plot

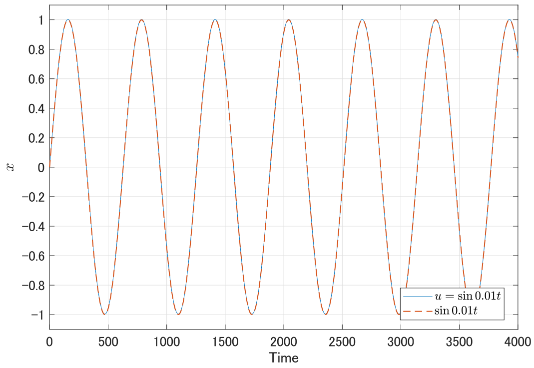

Fig. 5 Response for $u = \sin 0.01t$

7. CONCLUSION

In this paper, we have shown a novel characteristic in frequency analysis for r-homogeneous systems; every r-homogeneous control system with a sinusoidal input has an equivalent homogeneous closed-loop system, and frequency analysis is meaningful for r-homogeneous systems. Finally, we confirm the effectiveness of the frequency analysis of r-homogeneous systems by computer simulation.

Every finite-time stable r-homogeneous system has a negative homogeneous degree. As we showed in the example, an output signal is not attenuated relative to any small amplitude sinusoidal wave. This may be a key to the secret of the superior performance of r-homogeneous finite-time control in actual experiments; further analysis remains in the future.

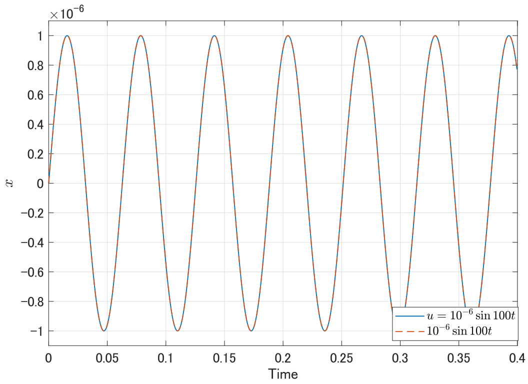

Fig. 6 Response for $u = 10^{-6} \sin 100t$

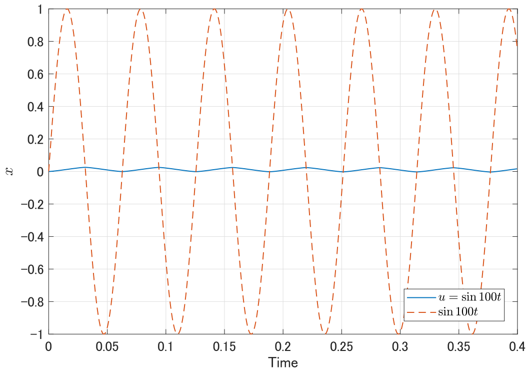

Fig. 7 Response for $u = \sin 100t$

We can view -20 dB/dec in Figs. 2–4, which is the same as linear systems. To answer the question of why -20 dB/dec is generated remains future work.

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