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A contact regulation control law for asymmetric damping to stabilize and robustify SIQRS model at endemic equilibrium

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Abstract: For mitigating the spread of infectious diseases which cannot be eliminated completely, this paper proposes a person-to-person contact regulation control law to stabilize and robustify the endemic equilibrium of SIQRS model subject to rate uncertainty of immunity, transmission, transportation, recovery, mortality, and inflows. Immunity waning creates a loop, which can induces undesired oscillations and makes nonlinear stability analysis not only harder but also prone to fatal conservativeness. To avoid such conservativeness, this paper designs feedback contact regulation adding asymmetric damping instead of cancelling nonlinearities. It renders the closed loop asymmetrically dissipative in terms of a logarithmic storage function serving as a control Lyapunov function. The function conforms to variables positivity and allows the regulation control to be combined with vaccination and isolation to achieve input-to-state stability.

Keywords: Infectious diseases; Endemic equilibria; Waning immunity. Lyapunov function; Asymmetric dissipativity;

1. INTRODUCTION

The central nonlinearity of disease models is bilineary [1, 4, 15, 16]. It makes the disease models admit two equilibria. The disease-free equilibrium, in which the population of infected individuals is zero, represents disease elimination. For diseases of high transmission rate, another equilibrium called the endemic equilibrium emerges. In the control literature, the disease-free equilibrium has been extensively studied due to its mathematical simplicity. By contrast, the endemic equilibrium prevents Lyapunov functions from being composed of component Lyapunov functions via summation since their time-derivative of cannot be strictly negative [6, 13]. As long as the endemic equilibrium is of interest, no stability regions can penetrate the zero hyperplane associated with the unstable disease-free equilibrium. A Lyapunov function can be effective only if its sublevel sets fit the asymmetry of the semi-infinite state space.

Logarithms are sometimes used in mathematical biology to obtain sublevel sets fitting the semi-infinite spaces [3, 17]. To design feedback control laws mitigating infectious diseases spread in the presence of susceptible inflow uncertainty, logarithmic Lyapunov functions were employed in the framework of input-to-state stability (ISS) under the assumption of persistent immunity [7, 8, 14, 21]. Those ISS properties were restricted to small state space regions which can be expanded at the price of unbounded increase of control input. Recently, immunity waning was studied in [11]. However, that initial achievement was only qualitative in the sense that the estimate of the admissible waning rate was impractically very small.

Aiming at quantitively useful treatment of wanning immunity, this paper proposes to add asymmetric damping by person-to-person contact regulation in an asymmetric dissipativity argument. It qualifies a logarithmic storage function as a control Lyapunov function leading to input-to-state stabilization of SIQRS model [18] with parameter uncertainties in the presence of an immunity waning rate considerably larger than that allowed in [11].

† Hiroshi Ito is the presenter of this paper.

2. SIQRS MODEL AND SETUP

Consider the dynamics of four population variables

$$\dot{S}(t) = -(\mu+\rho+u_2(t))S(t) - (\beta+u_1(t))S(t)I(t) + \eta R(t) + \underline{B} + r_1(t), \quad \text{(1a)}$$

$$\dot{I}(t) = (\beta+u_1(t))S(t)I(t) - (\gamma+\nu+u_3(t)+\mu)I(t) + r_2(t), \quad \text{(1b)}$$

$$\dot{Q}(t) = (\nu+u_3(t))I(t) - (\tau+\mu)Q(t) + r_3(t), \quad \text{(1c)}$$

$$\dot{R}(t) = (\rho+u_2(t))S(t) + \gamma I(t) + \tau Q(t) - (\mu+\eta)R(t) + r_4(t). \quad \text{(1d)}$$

The set of four equations (1) with $u_1 = u_2 = u_3 = 0$ and $r_1 = r_2 = r_3 = r_4 = 0$ is called the SIQRS model for $\eta > 0$ and the SIQR model for $\eta = 0$ [2, 5, 15, 18-20]. The variable S is the number of susceptible individuals in the geographical region of interest. The variable I is the number of infected individuals. This paper considers diseases among humans. The variable Q is the number of isolated individuals after infection, while R is the number of individuals recovered with immunity. The positive parameters γ, μ, and B are the recover rate, the mortality rate, and the inflow of immigrants and newborns, respectively. The non-negative parameters ρ, ν, and η are the nominal vaccination rate, the nominal isolation rate, and the rate of immunity waning, respectively. The positive parameter β denotes the disease transmission rate through person-to-person contact. The four state variables are packed into $x = (S, I, Q, R)$. We also use $u = (u_1, u_2, u_3)$ and $r = (r_1, r_2, r_3, r_4)$. This paper writes (1) compactly as $\dot{x} = f(x, u, r)$. The initial state is any $x(0) = (S(0), I(0), Q(0), R(0)) \in (0, \infty)^4$. The target of interest is the equilibrium $x_\star = (S_\star, I_\star, Q_\star, R_\star) \in (0, \infty)^4$ of (1) with $u = 0$ and $r = 0$. It is called the endemic equilibrium. From (1) with $u = 0$ and $r = 0$, it is calculated as

$$S_\star = \frac{\gamma+\mu+\nu}{\beta}, \quad I_\star = \frac{\tau+\mu}{\nu} Q_\star \tag{2a}$$

$$Q_\star = \frac{\nu(\mu+\rho+\eta)(\gamma+\mu+\nu)}{\beta((\gamma+\mu+\nu)(\mu+\tau)+\eta(\tau+\mu+\nu))}(\mathcal{R}_0-1), \tag{2b}$$

$$R_\star = \frac{1}{\mu+\eta}\left[\rho S_\star + \left(\frac{\gamma(\tau+\mu)}{\nu}+\tau\right)Q_\star\right], \tag{2c}$$

where the basic reproduction number $\mathcal{R}_0$ is defined as

$$\mathcal{R}_0 = \frac{B\beta(\mu+\eta)}{\mu(\rho+\mu+\eta)(\gamma+\mu+\nu)}. \tag{3}$$

The endemic equilibrium emerges if and only if $\mathcal{R}_0 > 1$. This paper assumes $\mathcal{R}_0 > 1$. From (1b) it is seen that the SIQRS model always admits another equilibrium satisfying $I = 0$. It is called the disease-free equilibrium, which is impractical to target for serious infectious diseases. The variable $u_1(t)$ is the control input to be designed, which describes the operation of person-to-person contact by changing society activity levels such as social distancing, waring masks, outing regulations, and lockdown. The input variables $u_2(t)$ and $u_3(t)$ are the adjustment rate of vaccination and isolation, respectively. The piecewise continuous $r_1(t) \in (0,\infty)$ represents the uncertainty and the lack of precise knowledge of the inflow and the parameters μ, ρ, β, and η. Let $r_{\star,1} = B - \underline{B} > 0$ and $r_{\star,2} = r_{\star,3} = r_{\star,4} = 0$. The constant $\underline{B} \in (0, B)$ presents the assumption that the population is not completely dying out. The piecewise continuous $(r_2(t), r_3(t), r_4(t)) \in [0,\infty)^3$ represent inflows from neighboring regions and the lack of precise knowledge of parameters. Naturally, the contribution of neighboring regions to r_3 is expected to be very small.

Define the deviation vector $\tilde{x} = x - x_\star$. For example, its first and second components are $\tilde{x}_1 = x_1 - x_{\star,1} = S - S_\star = \tilde{S}$ and $\tilde{x}_2 = x_2 - x_{\star,2} = I - I_\star = \tilde{I}$. The vector $\tilde{x}(t)$ evolves in the semi-infinite space $(-S_\star,\infty)\times(-I_\star,\infty)\times(-Q_\star,\infty)\times(-R_\star,\infty)$. To deal with semi-infinite spaces, given $z_* = (z_{*,1}, z_{*,2}, \ldots, z_{*,m}) \in [0,\infty)^m$, let $\mathbb{R}[z_*]$ denote the set $(-z_{*,1},\infty)\times(-z_{*,2},\infty)\times\cdots\times(-z_{*,m},\infty)\cup[0,\infty)^m$. A function $\Phi : \mathbb{R}[z_\star] \to [0,\infty)$ is said to belong to $\mathbf{M}[z_\star]$ and we write $\Phi \in \mathbf{M}[z_\star]$ if there exist a bijective increasing continuous function $\phi_i : (-z_{\star,i},\infty)\cup[0,\infty) \to \mathcal{Q}_i$ satisfying $\phi_i(0) = 0$, and a positive definite and radially unbounded continuous function $\phi : \mathbb{R}^m \to [0,\infty)$ such that

$$\Phi(\tilde{y}) = \phi((\phi_1(\tilde{y}_1), \phi_2(\tilde{y}_2), \ldots, \phi_m(\tilde{y}_m))), \tag{4}$$

where $\mathcal{Q}_i = \mathbb{R}$ if $z_{*,i} > 0$, otherwise $\mathcal{Q}_i = [0,\infty)$. The reader can refer to [9] for properties of $\mathbf{M}[z_\star]$. The following defines ISS on semi-infinite spaces [11, 21].

Definition 1: System (1) is said to be input-to-state stable (ISS) with positivity from r if a unique solution $x(t)$ to (1) exists for all $t \in [0,\infty)$, and there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}\cup\{0\}$, $\Phi \in \mathbf{M}[x_\star]$, and $\Psi \in \mathbf{M}[r_\star]$ such that

$$\Phi(\tilde{x}(t)) \le \beta(\Phi(\tilde{x}(0)), t) + \gamma\left(\sup_{\tau\in[0,t]} \Psi(\tilde{r}(\tau))\right) \tag{5}$$

holds for all $t \in [0,\infty)$ and for any $x(0) \in (0,\infty)^n$. If $\gamma = 0$, system (1) is said to be globally asymptotically stable with positivity.

The standard phrase "the equilibrium $x_\star$" is replaced by "system (1)" since the property is global. A function V in the next proposition is called an ISS Lyapunov function [11, 22]. It is called a Lyapunov function if $\sigma = 0$.

Proposition 1: If there exist α, $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}\cup\{0\}$, $\Phi \in \mathbf{M}[x_\star]$, and $\Psi \in \mathbf{M}[r_\star]$, and a continuously differentiable function $V : \mathbb{R}[x_\star] \to [0,\infty)$ such that

$$\forall \tilde{x} \in \mathbb{R}[x_\star] \quad \underline{\alpha}(\Phi(\tilde{x})) \le V(\tilde{x}) \le \overline{\alpha}(\Phi(\tilde{x})), \tag{6}$$

$$\forall (\tilde{x},\tilde{r}) \in \mathbb{R}[x_\star]\times\mathbb{R}[r_\star]$$
$$\frac{\partial V}{\partial \tilde{x}} F(x,r) \le -\alpha(V(\tilde{x}))) + \sigma(\Psi(\tilde{r})), \tag{7}$$

then system (1) is ISS with positivity from r. $\alpha \in \mathcal{P}$ can replace $\alpha \in \mathcal{K}_\infty$ if $\liminf_{s\to\infty}\alpha(s) \ge \lim_{s\to\infty}\sigma(s)$.

3. ASYMPTOTIC STABILIZATION

This paper uses the function $L(s_\star, \tilde{s}) = \tilde{s} - s_\star \ln(1 + \tilde{s}/s_\star)$ for $s_\star > 0$ and $\tilde{s} > -s_\star$ and emplys

$$V(\tilde{x}) = \sum_{i=1}^{4} \lambda_i L(x_{\star,i}, \tilde{x}_i)$$
$$=: V_S(\tilde{S}) + V_I(\tilde{I}) + V_Q(\tilde{Q}) + V_R(\tilde{R}). \tag{8}$$

The coefficients $\lambda_i > 0$, $i = 1, 2, 3, 4$, have yet to be determined. This paper proposes the feedback control law

$$u_1 = K_1 \max\left\{0, M_1(\tilde{S},\tilde{I})\right\}\left(\frac{\tilde{S}}{S} - \frac{\tilde{I}}{I}\right), \tag{9}$$

$$M_1(\tilde{S},\tilde{I}) = \frac{l\dfrac{\tilde{I}^2}{I} - \dfrac{(\delta+1)m}{2}\dfrac{\tilde{S}^2}{S}}{SI\left(\dfrac{\tilde{S}}{S} - \dfrac{\tilde{I}}{I}\right)^2}, \tag{10}$$

where $K_1 > 0$, $l > 0$, $m > 0$, and $\delta > 0$ have yet to be determined. The following theorem demonstrates how the above control law adds damping to the infection dynamics in terms of asymmetric dissipativity, which qualifies V in (8) as a Lyapunov function for (1).

Theorem 1: Assume that $\eta > 0$ and

$$\eta\beta(\gamma(\tau+\mu)+\nu\tau)I_\star R_\star < \rho(\tau+\mu)(\gamma+\mu+\nu)B \tag{11}$$

is satisfied. Let δ, l, m, and $\xi \in (0,1)$ defining (10) be

$$\delta = \frac{\eta(\gamma(\tau+\mu)+\nu\tau)I_\star R_\star}{\xi\rho(\tau+\mu)BS_\star}, \; l = \delta\frac{B}{I_\star}, \; m = \frac{B}{S_\star}, \tag{12}$$

$$\eta\beta(\gamma(\tau+\mu)+\nu\tau)I_\star R_\star < \xi\rho(\tau+\mu)(\gamma+\mu+\nu)B. \tag{13}$$

Then for any $K_1 \geq 1$, the control law (9) is locally Lipschitz on $\mathbb{R}[(S_\star, I_\star)]$ and guarantees that the SIQRS model (1) with $r = 0$ is globally asymptotically stable with positivity. Furthermore, a Lyapunov function is given by (8).

Proof: Let $\mu_1 > 0$ and $\mu_2 > 0$ be defined with

$$0 = -\mu_2 S_\star + \eta R_\star, \quad 0 = -\mu_1 S_\star + B. \tag{14}$$

Their existence are guaranteed by $\eta > 0$ and $B > 0$. Due to $S_\star$ in (2a), assumption (13) guarantees

$$\frac{\gamma(\tau+\mu)+\nu\tau}{\xi\nu\tau} \cdot \frac{\nu\tau\mu_2 I_\star}{\rho(\tau+\mu)B} = \delta < 1. \tag{15}$$

It is verified that in the case of $\tilde{I}/I = \tilde{S}/S$, we have

$$\frac{\tilde{I}^2}{I} = \frac{\tilde{I}\tilde{S}}{S} = \frac{I_\star}{S_\star} \cdot \frac{\tilde{S}^2}{S}.$$

Hence, we have

$$l\frac{\tilde{I}^2}{I} - \frac{(\delta+1)B}{2S_\star}\frac{\tilde{S}^2}{S} = \left(\frac{lI_\star}{S_\star} - \frac{(\delta+1)B}{2S_\star}\right)\frac{\tilde{S}^2}{S} = -\frac{1-\delta}{2}\frac{B}{S_\star}\frac{\tilde{S}^2}{S}$$

for $S \in \mathbb{R}[S_\star]$ when $\tilde{I}/I = \tilde{S}/S$. Due to (15), we have $1-\delta > 0$. Therefore, when the denominator of M_1 in (10) is zero, its numerator is negative. It implies that M_1 is locally Lipschitz in $(\tilde{S}, \tilde{I})$ on $\mathbb{R}[(S_\star, I_\star)]$, and it satisfies $\lim_{\tilde{S}\to 0} M_1(\tilde{S}, \tilde{I}) = 0$ along $\tilde{I}/I = \tilde{S}/S$. The same properties hold for u as a function of $(\tilde{S}, \tilde{I})$.

The stationary equations associated with (1) are

$$0 = -(\mu+\rho)S_\star - \beta S_\star I_\star + \eta R_\star + B, \tag{16a}$$
$$0 = \beta S_\star I_\star - (\gamma+\nu+\mu)I_\star, \tag{16b}$$
$$0 = \nu I_\star - (\tau+\mu)Q_\star, \tag{16c}$$
$$0 = \rho S_\star + \gamma I_\star + \tau Q_\star - (\mu+\eta)R_\star \tag{16d}$$

for $r = 0$. Define $\mu_i > 0$ for $i = 3.4, 5$ as

$$0 = \rho S_\star - \mu_4 R_\star, \quad 0 = \gamma I_\star - \mu_3 R_\star, \quad 0 = \tau Q_\star - \mu_5 R_\star.$$

Equations (16a) and (16d) hold with $\mu + \rho + \beta I_\star = \mu_1 + \mu_2$ and $\mu + \eta = \mu_3 + \mu_4 + \mu_5$. Write (1) for $r = 0$ as

$$\begin{aligned}
\dot{S} &= f_1(S, I, u_1) + f_2(S, R), \\
\dot{I} &= f_6(S, I, u_1), \\
\dot{Q} &= f_7(I, Q), \\
\dot{R} &= f_3(I, R) + f_4(S, R) + f_5(Q, R),
\end{aligned}$$

where

$$\begin{aligned}
&f_1(S, I, u_1) = -(\mu_1 - \beta I_\star)S - (\beta + u_1)SI + B, \\
&f_2(S, R) = -\mu_2 S + \eta R, \\
&f_6(S, I, u_1) = (\beta + u_1)SI - (\gamma+\nu+\mu)I, \\
&f_7(I, Q) = \nu I - (\tau+\mu)Q, \\
&f_3(I, R) = \gamma I - \mu_3 R, \quad f_4(S, R) = \rho S - \mu_4 R, \\
&f_5(Q, R) = \tau Q - \mu_5 R.
\end{aligned}$$

Let $\lambda_1 = \lambda_2 = 1$ for (8). As shown in [11], the model (1) with $r = 0$ and the control law (9) satisfies

$$\frac{\partial V_S}{\partial \tilde{S}} f_1 + \frac{\partial V_I}{\partial \tilde{I}} f_6 = -\mu_1 \frac{\tilde{S}^2}{S} - \left(\frac{\tilde{S}}{S} - \frac{\tilde{I}}{I}\right) SIu. \tag{17}$$

On the other hand, from (10) and $K_1 \geq 1$ we obtain

$$\begin{aligned}
&-\left(\frac{\tilde{S}}{S} - \frac{\tilde{I}}{I}\right) SIu_1 \\
&= -K_1 \max\left\{0, M_1(\tilde{S}, \tilde{I})\right\} SI \left(\frac{\tilde{S}}{S} - \frac{\tilde{I}}{I}\right)^2 \\
&= -\max\left\{0, l\frac{\tilde{I}^2}{I} - \frac{(\delta+1)B}{2S_\star}\frac{\tilde{S}^2}{S}\right\} - \mathcal{E}_1(\tilde{S}, \tilde{I}) \\
&\leq -l\frac{\tilde{I}^2}{I} + \frac{(\delta+1)B}{2S_\star}\frac{\tilde{S}^2}{S} - \mathcal{E}_1(\tilde{S}, \tilde{I}),
\end{aligned} \tag{18}$$

where

$$\mathcal{E}_2(\tilde{S}, \tilde{I}) = (K_1 - 1)\max\left\{0, M_1(\tilde{S}, \tilde{I})\right\} SI \left(\frac{\tilde{S}}{S} - \frac{\tilde{I}}{I}\right)^2 \geq 0.$$

Sine $\mu_1 = B/S_\star$, with the help of the second inequality in (15), applying (18) to (17) yields $1 - \delta > 0$ and

$$\frac{\partial V_S}{\partial \tilde{S}} f_1 + \frac{\partial V_I}{\partial \tilde{I}} f_6 \leq -\frac{\mu_1(1-\delta)}{2}\frac{\tilde{S}^2}{S} - l\frac{\tilde{I}^2}{I} - \mathcal{E}_1(\tilde{S}, \tilde{I}) \tag{19}$$

for all $(\tilde{S}, \tilde{I}) \in \mathbb{R}[(S_\star, I_\star)]$. Due to $\xi \in (0, 1)$, for $h \in (0, 1)$, Theorem 10 in [12] yields $\alpha_3 \in \mathcal{K}_\infty$ satisfying

$$\begin{aligned}
\frac{\partial V_R}{\partial \tilde{R}} f_3 - (1-h)l\frac{\tilde{I}^2}{I} &= \lambda_4 \frac{\tilde{R}}{R}\left(-\mu_3\tilde{R} + \gamma\tilde{I}\right) - (1-h)l\frac{\tilde{I}^2}{I} \\
&\leq -\alpha_3(V_I(\tilde{I}) + V_R(\tilde{R}))
\end{aligned} \tag{20}$$

for $0 < \lambda_4 \leq \xi(1-h)l/\gamma$. The argument of Theorem 4 in [10] proves the existence of $\alpha_{24} \in \mathcal{P}$ such that

$$\begin{aligned}
&\frac{\partial V_S}{\partial \tilde{S}} f_2 + \frac{\partial V_R}{\partial \tilde{R}} f_4 = \frac{\tilde{S}}{S}\left[-\mu_2\tilde{S} + \eta\tilde{R}\right] \\
&\quad + \lambda_4 \frac{\tilde{R}}{R}\left[-\mu_4\tilde{R} + \rho\tilde{S}\right] \leq -\alpha_{24}(V_S(\tilde{S}) + V_R(\tilde{R}))
\end{aligned} \tag{21}$$

for $\lambda_4 = \mu_2/\rho$. Due to $\xi \in (0, 1)$, Theorem 11 in [12] yields the existence of $\alpha_{75} \in \mathcal{K}_\infty$ such that

$$\begin{aligned}
&\frac{\partial V_Q}{\partial \tilde{Q}} f_7 + \frac{\partial V_R}{\partial \tilde{R}} f_5 - hl\frac{\tilde{I}^2}{I} \\
&= \lambda_3 \frac{\tilde{Q}}{Q}\left[-(\tau+\mu)\tilde{Q} + \nu\tilde{I}\right] + \lambda_4 \frac{\tilde{R}}{R}\left[-\mu_5\tilde{R} + \tau\tilde{Q}\right] - hl\frac{\tilde{I}^2}{I} \\
&\leq -\alpha_{75}(V_I(\tilde{I}) + V_Q(\tilde{Q}) + V_R(\tilde{R})))
\end{aligned} \tag{22}$$

with $\lambda_3 = \xi hl/\nu$ and $\lambda_4 = \lambda_3(\tau+\mu)/\tau$. Define

$$h = \frac{\nu\tau}{\gamma(\tau+\mu)+\nu\tau}, \tag{23}$$

which satisfies $h \in (0,1)$. The first equality in (15) gives

$$\frac{\xi hl(\tau+\mu)}{\nu\tau} = \frac{\mu_2}{\rho}.$$

The property

$$h = \frac{\frac{\nu\tau}{\gamma(\tau+\mu)}}{1+\frac{\nu\tau}{\gamma(\tau+\mu)}}$$

implied by (23) is identical with

$$\frac{\xi hl(\tau+\mu)}{\nu\tau} = \frac{\xi(1-h)l}{\gamma}. \tag{24}$$

Hence, $h \in (0,1)$ given by (23) achieves (20), (21), and (22) simultaneously with $\lambda_3 = \xi hl/\nu$ and $\lambda_4 = \lambda_3(\tau+\mu)/\tau$. Summing up (19), (20), (21), and (22) yields the existence of $\alpha \in \mathcal{K}_\infty$ and $\sigma = 0$ in Proposition 1. ■

If the immunity is persistent, the following is proved.

Theorem 2: Assume that $\eta = 0$. Define (10) with

$$l = \delta(\mu+\beta I_\star)\frac{S_\star}{I_\star}, \quad m = \mu + \beta I_\star \tag{25}$$

for any $\delta \in (0,1)$. Then for any $K_1 \geq 1$, the control (9) is locally Lipschitz on $\mathbb{R}[(S_\star, I_\star)]$ and guarantees that the SIQRS model (1) is globally asymptotically stable with positivity. Furthermore, (8) is a Lyapunov function.

Proof: Let $\mu_2 = 0$ and define $\mu_i > 0$ for $i = 1, 3.4, 5$ as

$$0 = -\mu_1 S_\star + B = \rho S_\star - \mu_4 R_\star = \gamma I_\star - \mu_3 R_\star = \tau Q_\star - \mu_5 R_\star.$$

Then equations (16a) and (16d) are satisfied with $\mu+\rho+\beta I_\star = \mu_1+\mu_2$ and $\mu+\eta = \mu_3+\mu_4+\mu_5$ for $\eta = 0$. We have $f_2 = 0$. Furthermore, (19) and (21) are replaced by

$$\begin{aligned}\frac{\partial V_S}{\partial \tilde{S}}f_1 + \frac{\partial V_I}{\partial \tilde{I}}f_6 \leq & -\frac{(\mu+\beta I_\star)(1-\delta)}{2}\frac{\tilde{S}^2}{S} - l\frac{\tilde{I}^2}{I} \\ & - \mathcal{E}_1(\tilde{S},\tilde{I}) - \rho\frac{\tilde{S}^2}{S}, \\ -\rho\frac{\tilde{S}^2}{S} + \frac{\partial V_R}{\partial \tilde{R}}f_4 = & -\rho\frac{\tilde{S}^2}{S} + \lambda_4\frac{\tilde{R}}{R}\left[-\mu_4\tilde{R} + \rho\tilde{S}\right] \\ \leq & -\alpha_{24}(V_S(\tilde{S}) + V_R(\tilde{R}))\end{aligned}$$

with $\alpha_{24} \in \mathcal{P}$ for any $0 < \lambda_4 \leq 1$. Hence, for any $\xi \in (0,1)$, there exists $h \in (0,1)$ such that

$$\lambda_4 = \frac{\xi hl(\tau+\mu)}{\nu\tau} \leq \min\left\{1, \frac{\xi(1-h)l}{\gamma}\right\}.$$

$\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \xi hl/\nu$ establishes the claim. ■

In contrast to the existence result reported in [11], the condition (11) imposed on η is explicit. Using $I_\star$ and $R_\star$ given explicitly in (2), we arrive at the following.

Theorem 3: Assume that $\mathcal{R}_0 > 1$ for $\eta = 0$. Given any $K_1 \geq 1$, there exists $\overline{\eta} > 0$ such that for each $\eta \in [0, \overline{\eta})$, the SIQRS model (1) with the control (9) is globally asymptotically stable with positivity. Moreover, the left side of (11) is bounded from above by a class $\mathcal{K}$ function of η.

Proof: $\mathcal{R}_0 > 1$ for $\eta = 0$ implies that $\mathcal{R}_0$ in (3) is larger than 1 for all $\eta \geq 0$ and increasing in η. The expression (2) gives $0 = \lim_{\eta\to 0^+} \eta I_\star R_\star < \lim_{\eta\to\infty} \eta I_\star R_\star < \infty$, and $\forall\eta > 0\ \eta I_\star R_\star > 0$. Since

$$Q_\star \leq \max\left\{1, \frac{(\tau+\mu+\nu)(\mu+\rho)}{(\gamma+\mu+\nu)(\mu+\tau)}\right\}\frac{\nu(\gamma+\mu+\nu)}{\beta(\tau+\mu+\nu)}(\mathcal{R}_0 - 1),$$

$Q_\star$ is bounded from above by an increasing continuous positive-valued function of η. Due to (2), $\eta R_\star$ is bounded from above by a class $\mathcal{K}$ function of η. Hence, $\eta I_\star R_\star$ admits a class $\mathcal{K}$ upper bound in η. ■

4. INPUT-TO-STATE STABILIZATION

Using the maximum vaccination rate $\overline{\rho} \in (0, \infty]$ and the maximum isolation rate $\overline{\nu} \in (0, \infty]$, define

$$\xi_2(s) = \max\{-\rho, \min\{\overline{\rho} - \rho, s\}\}, \tag{26}$$

$$\xi_3(s) = \max\{-\nu, \min\{\overline{\nu} - \nu, s\}\} \tag{27}$$

for $s \in \mathbb{R}$. Consider the feedback laws

$$u_2 = \xi_2\left(K_2(x)S\left(\frac{\tilde{S}}{S} - \lambda_4\frac{\tilde{R}}{R}\right)\right), \tag{28}$$

$$u_3 = \xi_3\left(K_3(x)I\left(\frac{\tilde{I}}{I} - \lambda_3\frac{\tilde{Q}}{Q}\right)\right) \tag{29}$$

with any continuous functions

$$\forall x \in (0,\infty)^4 \quad K_2(x) \geq 0, \quad K_3(x) \geq 0. \tag{30}$$

These control laws of vaccination and isolation add robustness to the SIQRS model (1) for the disturbances r.

Theorem 4: Assume that $\eta > 0$ and (11) are satisfied. Defining (10) with δ, l, m, and $\xi \in (0,1)$ given by (12) and (13). Then for any constant $K_1 \geq 1$ and continuous functions K_2 and K_3 satisfying (30), the control laws u_1, u_2 and u_3 given by (9), (28), and (29) guarantee that the SIQRS model (1) is ISS with positivity from r. Furthermore, an ISS Lyapunov function is given by V in (8).

Proof: Due to (30), u_2 and u_3 in (28) and (29) satisfy

$$\lambda_1\frac{\tilde{S}}{S}(-Su_2) + \lambda_4\frac{\tilde{R}}{R}Su_2 \leq 0, \ \lambda_2\frac{\tilde{I}}{I}(-Iu_3) + \lambda_3\frac{\tilde{Q}}{Q}Iu_3 \leq 0.$$

Since $\partial L(x_{\star,i}, \tilde{x}_i)/\partial\tilde{x}_i = \tilde{x}_i/x_i$, by virtue of Theorem 1, applying $r_2 = \tilde{r}_2$, $r_3 = \tilde{r}_3$ $r_4 = \tilde{r}_4$, and Lemma 16 in [11] to the equations of (1) with (8) yields (7). ■

To represent the minimum and maximum of the adjustable society activity level, let β_b and β_u be such that $0 < \beta_b < \beta < \beta_u$. Define the limiter

$$\xi_1(s) = \max\{\beta_b - \beta, \min\{\beta_u - \beta, s\}\}. \tag{31}$$

If the control law (9) is replaced by

$$u_1 = \xi_1(\text{Eq.}(9)), \tag{32}$$

achieved asymptotic stability and ISS of (1) are restricted to a non-empty forward invariant subset of $(0,\infty)^4$, which is a sublevel set of V as in Corollary 4 of [11].

5. SIMULATIONS

This section illustrates the effectiveness of the proposed controller consisting of (32), (28), and (29) through simulations with $\beta = 0.315/217$, $\gamma = 0.03$, and $\tau = 0.04$ reported in [2] for COVID-19 in Brazil, 2020 and the parameters $B = 7900 \times 10^{-6}$ and $\mu = 0.000018$ computed from immigrants, newborns, and natural death reported in [23]. The unit of population and time is millions and days, respectively. We use the initial condition $X(0) = (216.9898, 0.01, 0.0001, 0.0001)$ and

$$r_1(t) = -790 \times 10^{-6} \cos(\pi t/150),$$
$$r_2(t) = \max\{0, 1000 \times 10^{-6} \sin(\pi t/75)\},$$
$$r_3(t) = \max\{0, 1000 \times 10^{-6} \cos(\pi t/75)\},$$
$$r_4(t) = \max\{0, -1000 \times 10^{-6} \sin(\pi t/75)\}$$

used in [11]. This paper uses the immunity waning rate

$$\eta = 10^{-5}. \qquad (33)$$

This is much larger than $\eta = 10^{-9}$ tackled in [11] for that established region-restricted ISS. For (33), Fig. 1 plots the state transition of the model (1) with no mitigating inputs. i.e., $\rho = \nu = 0$ and $u_1 = u_2 = u_3 = 0$. For feedback control, let the nominal vaccination and isolation rates be $\rho = 0.0004$, $\nu = 0.002$, which results in $\mathcal{R}_0 = 1.3017$. The corresponding target endemic equilibrium is $S_\star = 22.057$, $I_\star = 0.08893$, $Q_\star = 0.004445$, and $R_\star = 416.74$. Criterion (11) for the global closed-loop ISS in Theorem 4 is satisfied as $6.890 \times 10^{-10} < 4.049 \times 10^{-9}$. Figure 2 shows the state transition of the SIQRS model (1) controlled by (32), (28), and (29) with

$$K_1 = 1,\ K_2 = 0.001/S_\star,\ K_3 = 0.0001/I_\star,\ \overline{\nu} = 0.03,$$
$$\xi = 0.99999,\ \overline{\rho} = 0.003,\ \beta_b = 0.25\beta,\ \beta_u = 1.25\beta.$$

The comparison between Figs. 1 and 2 demonstrates the significant reduction of the peak of I. Figures 3, 4, and 5 are control inputs. The control input does not hit the limiter in (32). For another comparison, in Fig. 6, the state transition is plotted for no mitigation input except for vaccination at the maximum rate, i.e.,

$$u_1 = 0,\ \rho + u_2 = 0.003,\ \nu + u_3 = 0. \qquad (34)$$

The rate 0.003 is as large as administrating about $217 \times 0.003 \times 30$[days] $\simeq$ 195 millions shots in the first month period. Since (34) gives $\mathcal{R}_0 = 0.1964$, the endemic equilibrium disappears and the disease-free equilibrium is the only stationary point. Fig. 6 in which the peak of I remains very high shows that removing a disease has nothing to do with reducing its infection peak. Figure 2 illustrates how effective and important the isolation and contact regulations are for mitigating the disease spread.

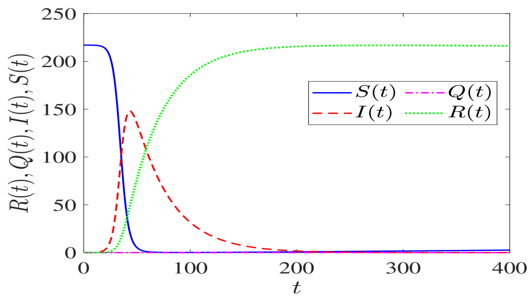

Fig. 1 State transition of (1) with no mitigating inputs.

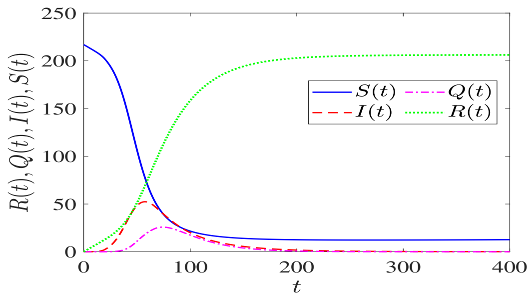

Fig. 2 State transition of (1) controlled with (32), (28), and (29).

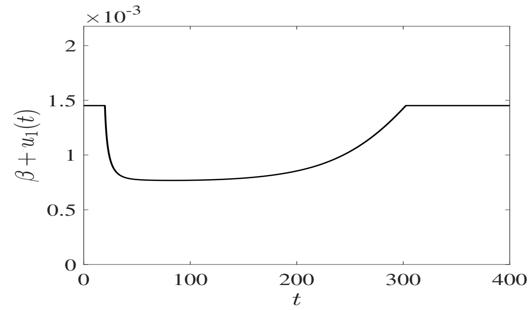

Fig. 3 Transmission with contact regulation $\beta + u_1$.

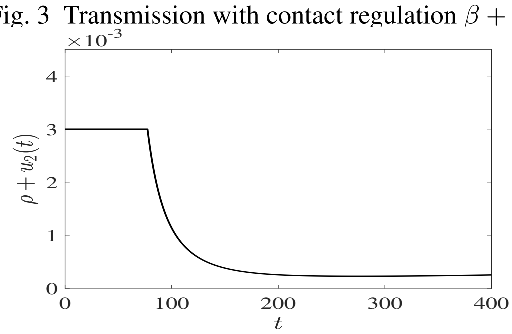

Fig. 4 Vaccination rate $\rho + u_2$.

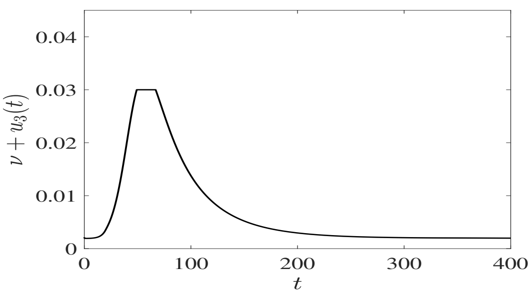

Fig. 5 Isolation rate $\nu + u_3$.

6. CONCLUSION

Using person-to-person contact regulation (9) that transforms the disease transmission dynamics to an asymmetric supply rate (19), this paper has proposed a gradient descent feedback control to robustly mitigate

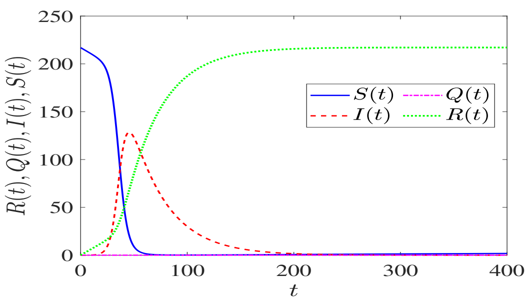

Fig. 6 State transition of (1) with constant vaccination (34) only.

infectious disease spread with a balanced combination of the contact regulation, isolation (29), and vaccination (28). The transformation allows us to treat the SIQRS model as an interconnection consisting of (19)-(22). It qualifies the logarithm function (8) as a control Lyapunov function to compose the gradient descent control.

The logarithmic function guarantees variables positivity and yielding sublevel sets compatible with the semi-infinite state space on which the stability and robustness guarantees should be expanded. Nevertheless, finding control laws that entitle the logarithm function as a control Lyapunov function had not been easy. This paper's result, which gives a a successful answer, contrasts sharply with the existing semi-global and regional methods [7, 8, 14] without waning immunity and with waning immunity [11]. Furthermore, this paper provides an explicit condition as in (11) on the admissible rate of waning immunity, which had been only the existence in [11].

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