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Frequency response shaping of type 2 output sensitivity for type 1 servo systems

Reiji Kawakami[1†], Kana Shikada[2], Noboru Sebe[3], and Masayuki Sato[4]

[1]Department of Creative Informatics, Kyushu Institute of Technology, Fukuoka, Japan
(E-mail: kawakami.reiji658@mail.kyutech.jp)
[2]Department of Aeronautics and Astronautics, Graduate School of Engineering, Kyoto University, Kyoto, Japan
(E-mail: shikada@kuaero.kyoto-u.ac.jp)
[3]Department of Intelligent and Control Systems, Kyushu Institute of Technology, Fukuoka, Japan
(E-mail: sebe@ics.kyutech.ac.jp)
[4]Faculty of Advanced Science and Technology, Kumamoto University, Kumamoto, Japan
(E-mail: sato.masayuki@mech.kumamoto-u.ac.jp)

Abstract: This paper proposes an H_∞ controller design method for two-degree-of-freedom (2-DOF) control systems with lower sensitivity characteristics in the low-frequency range similar to that achieved by disturbance observers. The slopes of the magnitude plots of the sensitivity functions achieved by disturbance observers are 40 [dB/dec], while the systems are servo systems for step signals. Previous research demonstrated that incorporating integral weighting for input disturbances could achieve such low-sensitivity characteristics. However, with that method, the dynamics of the plant affect the output sensitivity function in the evaluated transfer function. Therefore, it is difficult to shape the output sensitivity function precisely. This paper emphasizes the importance of directly shaping the output sensitivity functions and proposes to move the integral weight from the input side of the plant to the output side. This approach allows for precisely shaping sensitivity functions while maintaining servo characteristics. Numerical examples demonstrate the effectiveness of the proposed method.

Keywords: Servo systems, Low output sensitivity characteristics, H_∞ control, Robust control

1. INTRODUCTION

For control system design, it is important to consider the uncertainty of the plants. Uncertainties always exist in actual systems due to aging, environmental changes, and external interference. If such uncertainties exist, the possibility that the designed control system will not perform as expected increases. Therefore, the design must take into account the uncertainties of plants and maintain control performance.

Various robust control approaches have been developed to address the problem caused by uncertainty in control systems. One such approach is the H_∞ control. Based on the small gain theorem, the H_∞ control provides a systematic method for designing controllers that can maintain performance and stability despite uncertainties. The H_∞ control problem involves designing a controller based on the frequency response shaping. In the H_∞ control, the H_∞ norm of focused transfer functions, such as the sensitivity function with frequency weights, is adopted as an evaluation function to measure control performance quantitatively, and this evaluation function is optimized [1, 2]. For example, servo systems for step responses can be designed by evaluating the transfer function from the reference input to the error signal, i.e., the difference between the reference input and the controlled output, with an integral weight. Based on this, the mixed sensitivity problem is proposed [3, 4].

The sensitivity function denotes how the relative error of the control plant affects the relative error of the transfer function of the closed-loop system. Accordingly, the smaller the magnitude of the sensitivity function, the smaller the effect of uncertainties in that frequency range. For one-degree-of-freedom (1-DOF) control systems, the transfer function from the reference input to the error signal coincides with the sensitivity function. However, it does not coincide with the sensitivity function of two-degree-of-freedom (2-DOF) control systems. In the case of 2-DOF systems, the sensitivity function coincides with the transfer function from the output disturbance to the output, and it depends only on the feedback part of the controllers [5, 6].

In 2-DOF control systems, tracking characteristics and robust stability can be specified independently. In addition, the servo and sensitivity characteristics will be the same type in 2-DOF systems designed using ordinary design procedures, such as 2-DOF PID control design. (In this paper, the “types” of the servo and sensitivity characteristics are defined by the slope of their magnitude plots in the low-frequency range. If the characteristic is called “type n”, the slope of its magnitude plot is $20 \times n$ [dB/dec] in the low-frequency range.) However, in contrast to the ordinary design procedures, the disturbance observer [7] can specify the type of sensitivity functions independent of the type of servo characteristics. Therefore, even specifying type 1 servo characteristics, the disturbance observer can achieve type 2 sensitivity characteristics [8, 9].

The previous study [10] proposed a method to achieve type 2 low sensitivity characteristics while maintaining type 1 servo characteristics by H_∞ control. Specifically,

† Reiji Kawakami is the presenter of this paper.

the method applies an additional integral weight to the ordinary generalized plant for type 1 servo systems design at the input disturbance. Although this approach can attain characteristics similar to those of disturbance observers, this method does not evaluate the exact output sensitivity function. Therefore, it is difficult to shape the output sensitivity directly.

To solve the above problem, this paper proposes a method to design 2-DOF controllers that achieve the type 1 servo characteristics. This method directly shapes the type 2 sensitivity characteristics in H_∞ control; specifically, integral weights that evaluate the output disturbance are added. Furthermore, integral weights often cause numerical instability in the design calculations. To avoid this problem, we also propose a generalized plant configuration. Numerical examples demonstrate the achievable low-sensitivity characteristics of the 2-DOF servo system designed by the proposed method. The effectiveness of the proposed method is confirmed by comparing the sensitivity characteristics with those of previous studies where the input disturbance includes integral characteristics.

Notation: In general, signals are function of time t. However, for the sake of simplicity, we do not explicitly describe that they are functions of t.

2. PRELIMINARIES

2.1. Sensitivity function of 2-DOF control system [5, 6]

This section briefly reviews the definition of the output sensitivity function for multi-input and multi-output systems.

Figure 1 shows the general block diagram of a 2-DOF control system, where $P(s)$ is a given plant and a 2-DOF controller $C(s)$ consists of proper transfer function matrices $K_r(s)$ and $K_y(s)$. The signals r, u, y, and d_o denote the reference input, control input, output, and output disturbance, respectively. Let $P_m(s)$ be the nominal model of the plant, and suppose the plant perturbed with a multiplicative uncertainty coincides with the nominal model, i.e.,

$$P_m(s) = \big(I + \delta P(s)\big)P(s), \tag{1}$$

where $\delta P(s)$ denotes the multiplicative perturbation. Note that this definition of the multiplicative perturbation differs from the ordinary one, and is called the inverse multiplicative perturbation [11]. Similarly, let $T_m(s)$ and $T(s)$ be the transfer function matrices from r to y corresponding to $P_m(s)$ and $P(s)$, respectively. In addition, similarly define the multiplicative perturbation of the closed-loop transfer function $T(s)$ as $\delta T(s)$, i.e.,

$$T_m(s) = \big(I + \delta T(s)\big)T(s). \tag{2}$$

Then, the definition of the output sensitivity function is given as follows.

Definition 1. The output sensitivity function $S(s)$ of the closed-loop system shown in Fig. 1 is defined as

$$\delta T(s) = S(s)\,\delta P(s). \tag{3}$$

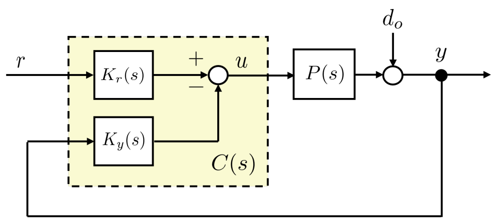


Fig. 1. General 2-DOF control system.

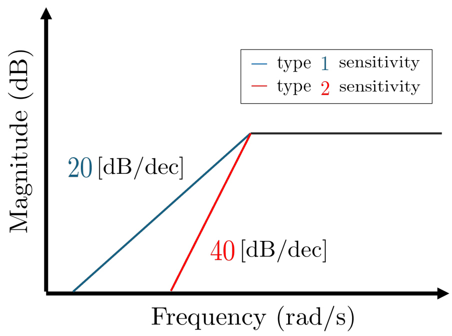


Fig. 2. Magnitude diagrams of sensitivity functions of types 1 and 2 servo systems.

Then, the following proposition gives the explicit form of the output sensitivity function $S(s)$.

Proposition 1.
The sensitivity function in Fig. 1 is given as follows:

$$S(s) = (I + P_m(s)K_y(s))^{-1}. \tag{4}$$

The output sensitivity function $S(s)$ represents how the transfer function matrix $T(s)$ of the closed-loop system varies compared to the transfer function matrix $P(s)$ of the plant. Therefore, the smaller the magnitude of the sensitivity function, the further the effect of the uncertainties in that frequency can be reduced. In the 2-DOF control system shown in Fig. 1, the sensitivity function $S(s)$ corresponds to the transfer function from d_o to y. Note that the sensitivity function of the 2-DOF control system is independent of $K_r(s)$ and is the same as that of the 1-DOF control system consisting of $K_y(s)$ and $P(s)$.

2.2. Servo and sensitivity characteristics

One of the important specifications in control system design is that the control outputs follow the step and ramp reference inputs without steady-state errors. The problem of designing to meet these specifications is called the servo system design problem. It is well known that control systems that follow step inputs have one integrator in control systems. This fact is called the internal model principle [12].

The blue line of Fig. 2 shows the magnitude diagram of the servo characteristics, i.e., the transfer function from the reference input to the error signal, of a servo system for step inputs. As shown in Fig. 2, the slope of the magnitude at the low-frequency range is 20 [dB/dec]. The magnitude diagram of the sensitivity function of 1-DOF servo systems has the same slope as the blue line in Fig. 2.

Similarly, the slope of the sensitivity function of 1-DOF servo systems for ramp inputs becomes 40 [dB/dec], as shown in the red line of Fig. 2. For 1-DOF servo systems, both types of servo and sensitivity characteristics are the same.

In 2-DOF servo systems, the types of servo and sensitivity characteristics may differ. However, in most design procedures, 2-DOF servo systems have the same types of servo and sensitivity characteristics. Servo systems with disturbance observers have type 2 sensitivity characteristics while maintaining the type 1 servo characteristics [8]. From the meaning of the sensitivity function, the effect of uncertainty is small at frequencies where the sensitivity function is small. This implies that servo systems with type 2 sensitivity suppress the variation of responses caused by uncertainties. (See Section 4.)

2.3. Previous design method [10]

A block diagram of the ordinary generalized plant used to design 2-DOF servo controllers with the H_∞ control is shown in Fig. 3. In Fig. 3, w_u is the weight for evaluating the magnitude of the control input, $W_{id}(s)$ and $W_{od}(s)$ are the weights for the input and output disturbances, respectively. It is assumed that $P(s)$ is proper and has no zeros at $s = 0$. This assumption is necessary to achieve the type 1 servo system. It is also assumed that $P(s)$ has no pole at $s = 0$. This assumption is technical and is for simplicity.

Generally, in servo system design by H_∞ control, it is necessary to consider the transfer function from the reference input r to the tracking error e, multiplied by the weight $W_s(s)$ [2]. For designing type 1 servo systems, this weight is typically specified as $W_s(s) = \frac{w_a s + w_b}{s}$. In Fig. 3, the integrator of $W_s(s)$ in embedded into the closed-loop system [13]. This weight specifies the servo characteristics, where w_a is a parameter that specifies vibration suppression performance and w_b specifies the rise time of the time response. The actual controller is shown as the dotted block in Fig. 3 and is composed of the servo integrators and $K(s)$, where $K(s)$ represents the part of the actual controller that is obtained by the design calculation. Then, the sensitivity function is shaped by finding a controller $K(s)$ such that the H_∞ norm of the system from r to z_1 is reduced. The servo systems with the controller designed by this method generally achieve the same type of sensitivity characteristics as their servo characteristics.

The previous study [10] focuses on the weight $W_{id}(s)$ that evaluates the input disturbance in the generalized plant shown in Fig. 3 to achieve the type 2 sensitivity characteristics. The weight $W_{id}(s)$ is often chosen as a constant weight considering all frequency ranges or white noise considering high-frequency ranges. The previous study considers a weight $W_{id}(s)$ to include integral characteristics as follows:

$$W_{id}(s) = \frac{w_{id1} s + w_{id2}}{s}. \quad (5)$$

In $W_{id}(s)$, there is a newly introduced tuning parameter w_{id2}. This tuning parameter specifies the degree of low sensitivity in the low-frequency range. Theoretically, applying the weight $W_{id}(s)$ in (5) enforces the resulting controller to include an additional integrator and achieves type 2 sensitivity characteristics while maintaining type 1 servo characteristics. However, the straightforward implementation of the weight $W_{id}(s)$ in (5) shown in Fig. 3 causes numerical problems. Accordingly, similar to the $W_s(s)$, the previous study proposed to embed the integrator of $W_{id}(s)$ into the closed-loop system as shown in Fig. 4.

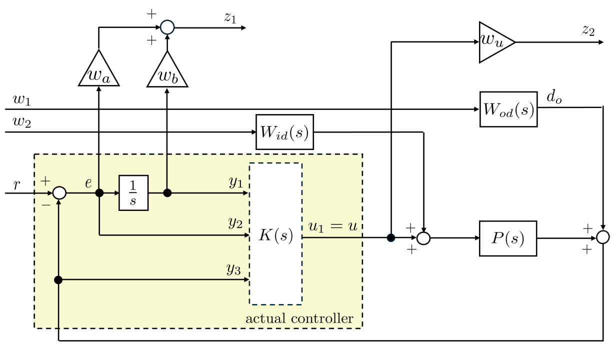


Fig. 3. Ordinary generalized plant for designing 2-DOF servo systems.

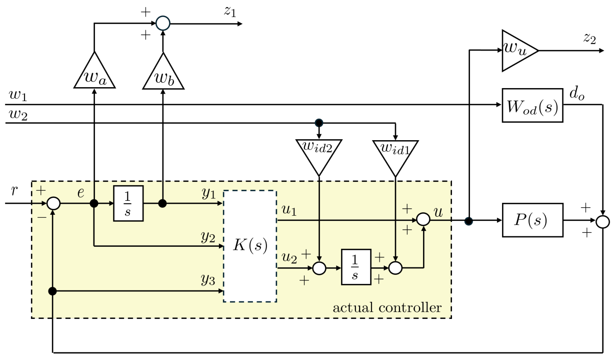


Fig. 4. Generalized plant for designing 2-DOF servo systems proposed in [10].

Remark.
The method proposed in [10] can also theoretically shape the output-side sensitivity function if we use $W_{id}(s)$ given below instead of equation (2).

$$W_{id}(s) = \frac{w_{id1} s + w_{id2}}{s} \cdot P^{-1}(s). \quad (6)$$

However, this weight may cause numerical problems, since it cancels the dynamics of $P(s)$, especially when performing LMI-based design calculations. Furthermore, in most cases, $P(s)$ is strictly proper, so a filtered inverse of $P(s)$ must be used. These issues are not easy to solve.

3. PROPOSED METHOD

This paper aims to design a 2-DOF controller by directly shaping the sensitivity function while maintaining the type 1 servo characteristics and achieving the type 2 sensitivity characteristics by H_∞ control theory. As is well known as the water bed effect, in most cases, the sensitivity function cannot be reduced over the entire frequency range. Therefore, if type 2 sensitivity is specified, the magnitude of the sensitivity becomes large in the middle-frequency range. There is a possibility that the time response may become oscillatory at the cost of reducing the response variability. In order to reduce this possibility, shaping the sensitivity function in the middle-frequency range is important.

A block diagram of the generalized plant typically used to design a 2-DOF controller with H_∞ control is shown in Fig. 3. This paper focuses on the weight $W_{od}(s)$ that evaluates the output disturbance in the generalized plant shown in Fig. 3. As described in Section 2.3, based on the H_∞ control theory, the previous study proposed to include integral characteristics in the weight $W_{id}(s)$ that evaluates the input disturbance. However, the objective function, i.e., the transfer function from w_2 to z_1, consists of not only the weights and output sensitivity but also the plant dynamics. Accordingly, the objective function differs from the exact output sensitivity function, and the previous method cannot directly shape it. This paper proposes to include integral characteristics in the weight $W_{od}(s)$ instead of the weight $W_{id}(s)$. However, there is a problem with this method. An integrator cannot be embedded into the closed-loop system as in the case of $W_{id}(s)$. This is because if an integrator is embedded in the output side of the plant, the error signal will be different from $r - y$, and the servo characteristics cannot be adequately specified. Instead of embedding an integrator in the output side of the plant, this paper proposes to add an integrator at the output side of the controller as in the previous study, as shown in Fig. 5 (a). Furthermore, since the integral characteristics of $W_{od}(s)$ could not be embedded in the closed-loop system, the following $\tilde{W}_{od}(s)$ with a slight modification of the integral characteristics is adopted as a weight to achieve a type 2 sensitivity function.

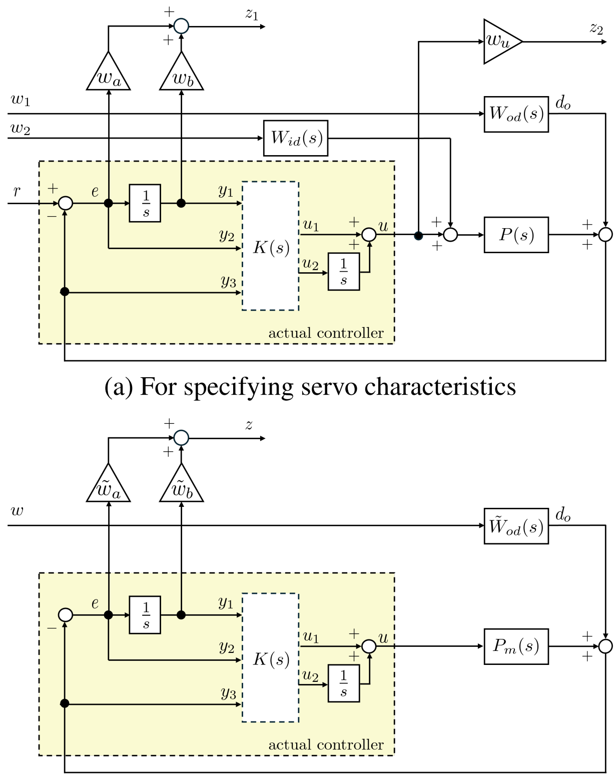
$$\tilde{W}_{od}(s) = \frac{\tilde{w}_{od1}s + \tilde{w}_{od2}}{s + \varepsilon}, \tag{7}$$

where ε is a parameter to make $\tilde{W}_{od}(s)$ stable and is chosen to be small, e.g., $10^{-2} \sim 10^{-3}$, with the consideration of numerical stability of design calculation.

With such an arrangement of weights and integrators, type 1 servo systems with type 2 sensitivity characteristics can theoretically be designed. However, the existence of the zeros of the closed-loop system at $s = 0$ produced by integrators and the pole of the weight at $s = -\varepsilon$ often causes numerical problems in the design calculations. Therefore, we proposed to design controllers based on the multi-model approach [14] using generalized plants that specify the servo characteristics and the sensitivity characteristics separately. Figures 5 (a) and 5 (b) show the generalized plants for specifying servo characteristics and for shaping output sensitivity characteristics, respectively. In other words, the proposed method designs a controller $K(s)$ by using two types of generalized plants. The generalized plant shown in Fig. 5 (a) only specifies the servo characteristics, and $W_{od}(s)$ does not include integral characteristics. On the other hand, the generalized plant shown in Fig. 5 (b) specifies the type 2 sensitivity characteristics and $\tilde{W}_{od}(s)$ in (7) is used. Then, by using Fig. 5 (b) to evaluate the transfer function from w to z, the magnitude of the output sensitivity function can be shaped smaller than that of $\gamma/\tilde{W}_e(s)$, where γ is the H_∞ norm achieved by the designed controller $K(s)$ and

$$\tilde{W}_e(s) = \frac{(\tilde{w}_a s + \tilde{w}_b)(\tilde{w}_{od1}s + \tilde{w}_{od2})}{s(s + \varepsilon)}. \tag{8}$$

With the generalized plant in Fig. 5 (b), we can specify the type 2 output sensitivity characteristic directly.



(a) For specifying servo characteristics

(b) For shaping sensitivity characteristics

Fig. 5. Proposed generalized plant for designing 2-DOF servo systems.

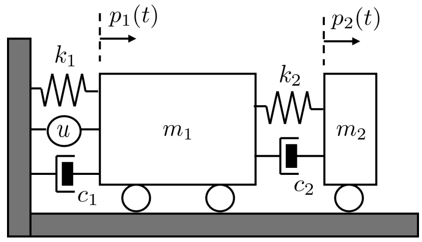


Fig. 6. Two-cart system.

4. NUMERICAL EXAMPLE

In this section, the effectiveness of the proposed method is demonstrated through a numerical example. In the numerical example, three 2-DOF controllers are designed by H_∞ control with the ordinary method, the previous design method, and the proposed method. The parameter that stabilizes the proposal weights (7) is set to ε = 10^{-2}. These controllers are designed by the MATLAB function hinfstruct() [15].

4.1. Problem settings

Let us consider a two-cart system shown in Fig. 6 as a plant for this numerical example, where the parameters m_1 [kg] and m_2 [kg] are masses of carts 1 and 2, k_1 [N/m] and k_2 [N/m] are spring constants, and c_1 [N s/m] and c_2 [N s/m] are viscous friction coefficients, respectively. The p_1[m] and p_2[m] are the positions of carts 1 and 2, respectively, and the measurement output y is the position of cart 2. Furthermore, friction with the floor surface is ignored. The control input u is the voltage applied to the actuator, and K_s times the voltage is applied to cart 1 as a force. This numerical example considers an uncertainty (gain fluctuation) at the control input. Let k_g represent the input gain, and it is represented by $k_g = 1 + \Delta$, where the gain fluctuation is denoted by Δ. Therefore, the nominal plant model $P_m(s)$ is given by $\Delta = 0$. The state-space representation of the plant $P(s)$ is given below.

$$\dot{x} = A_p x + B_p u, \tag{9a}$$

$$y = C_p x, \tag{9b}$$

where

$$A_p = \begin{bmatrix} O & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad B_p = \begin{bmatrix} O \\ k_g \times M^{-1}F \end{bmatrix},$$

$$C_p = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix},$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \quad F = \begin{bmatrix} K_s \\ 0 \end{bmatrix}.$$

The values of each physical parameter are $m_1 = 0.8$ [kg], $m_2 = 0.2$ [kg], $k_1 = 100$ [N/m], $k_2 = 300$ [N/m], $c_1 = 0.5$ [N s/m], $c_2 = 0.3$ [N s/m], and $K_s = 100$ [N/V].

4.2. Design specifications

The common weight parameters in Figs. 3, 4 and 5 (a) are specified as $(w_a, w_b, w_u) = (0.9, 0.5, 0.1)$. Then, the weights $W_{id}(s)$, $W_{od}(s)$, and $\tilde{W}_{od}(s)$ are specified as follows.

(i) Ordinary method with the generalized plant in Fig. 3.

$$W_{id}(s) = 0.01, \quad W_{od}(s) = 0.15.$$

(ii) Previous design method with the generalized plant in Fig. 4.

$$W_{id}(s) = \frac{0.01s + 7}{s}, \quad W_{od}(s) = 0.15.$$

(iii) Proposed design method with the generalized plant in Figs. 5 (a) and 5 (b).

$$W_{id}(s) = 0.01, \qquad W_{od}(s) = 0.15,$$

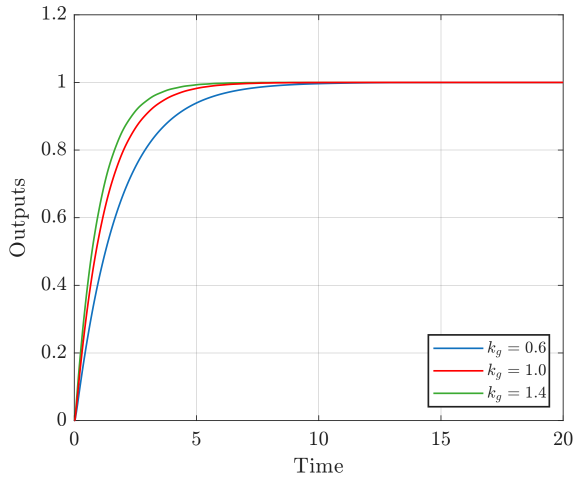


Fig. 7. Step responses achieved by ordinary method.

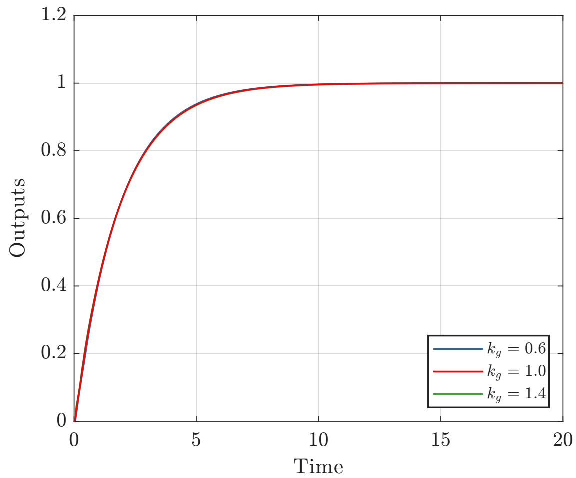


Fig. 8. Step responses achieved by previous design method.

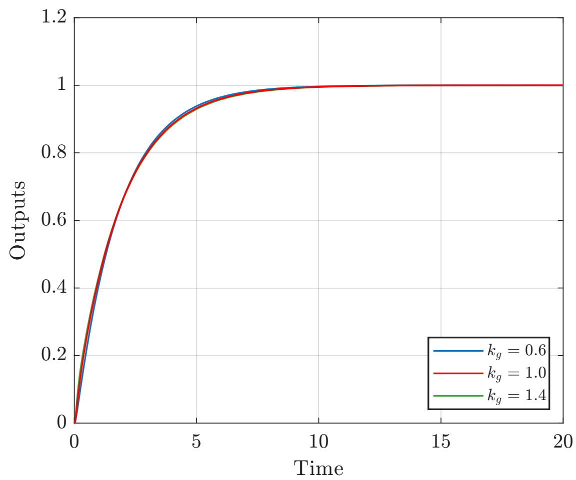


Fig. 9. Step responses achieved by proposed method.

$$\tilde{W}_{od}(s) = \frac{0.9s + 7}{s + 10^{-2}}, \quad (\tilde{w}_a, \tilde{w}_b) = (0.9, 0.5).$$

For each design specification, controllers $K(s)$ are designed by applying the multi-model approach [14]. In Figs. 3, 4, and 5 (a), three plant models are selected with $\Delta = 0, \pm 0.4$. In the ordinary and the previous design methods, controllers $K(s)$ are designed by minimizing the worst-case H_∞ norm of the transfer function matrices from $[r\, w_1\, w_2]^{\mathrm{T}}$ to $[z_1\, z_2]^{\mathrm{T}}$ of Figs. 3 and 4 corresponding to the three plant models by the MATLAB function hinfstruct() [15]. In the proposed method, a controller $K(s)$ is designed by minimizing the worst-case H_∞ norm of the transfer function matrices from $[r\, w_1\, w_2]^{\mathrm{T}}$ to $[z_1\, z_2]^{\mathrm{T}}$ of Fig. 5 (a) corresponding to the three plant models and the transfer function from w to z of Fig. 5 (b) by hinfstruct().

4.3. Design results

The achieved H_∞ norms in designs (i), (ii), and (iii) are $\gamma_1 = 0.9300$, $\gamma_2 = 0.9895$, and $\gamma_3 = 0.9342$, respectively. The parameters of the designed controllers are shown in Appendix.

The step responses achieved by the controllers in cases (i), (ii), and (iii) are shown in Figs. 7, 8 and 9, respectively. As shown in Fig. 7, the servo system consists of the controller designed with the constant weight $W_{od}(s)$ has varied responses against gain fluctuations. On the other hand, as shown in Figs. 8 and 9, if the weight $W_{id}(s)$ or $W_{od}(s)$ includes integral characteristics, the variation of the outputs is suppressed compared to the case (i) shown in Fig. 7.

Figure 10 shows the magnitude plots of the transfer function from the reference input r to the tracking error e in the case of (i), (ii), and (iii). In each case, it is confirmed that the servo characteristics have a slope of 20 [dB/dec] in the low-frequency range. This implies that all the closed-loop systems are type 1 servo systems.

Figure 11 shows the magnitude plots of the transfer function from the output disturbance d_o to the output y_3 (output sensitivity function). Also, the dashed line in Fig. 11 represents the magnitude plot of $\gamma_3/\tilde{W}_e(s)$. With the ordinary weight, the gain of the sensitivity function has a slope of 20 [dB/dec] in the low-frequency range, a characteristic of a standard type 1 servo system. With the previous and the proposed weights, the magnitude plots of the sensitivity functions have slopes of 40 [dB/dec] in the low-frequency range. Unlike the servo characteristics shown in Fig. 10, the proposed method achieves the type 2 sensitivity characteristics similar to the previous study method. The previous design method significantly reduces the sensitivity in the low-frequency range, but it can be confirmed that a larger peak than any other method appears in the range of approximately 20∼80 [rad/s]. However, the proposed method shapes the output sensitivity function below $\gamma_3/\tilde{W}_e(s)$, and the peak that appeared in the previous design method is suppressed. If $\gamma_3/\tilde{W}_e(s)$ is specified larger in the high-frequency band, the magnitude of the sensitivity function can be reduced more in the low-frequency band. Therefore, the proposed method can be used to shape the sensitivity function as desired. It is confirmed that by adding integral characteristics to the weight $W_{od}(s)$ for the output disturbance, it is possible to shape the output sensitivity functions as type 2 and design type 1 servo systems.

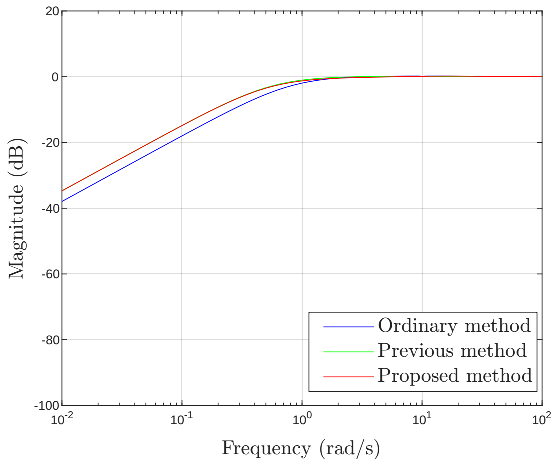


Fig. 10. Magnitude plots of nominal transfer functions from r to e.

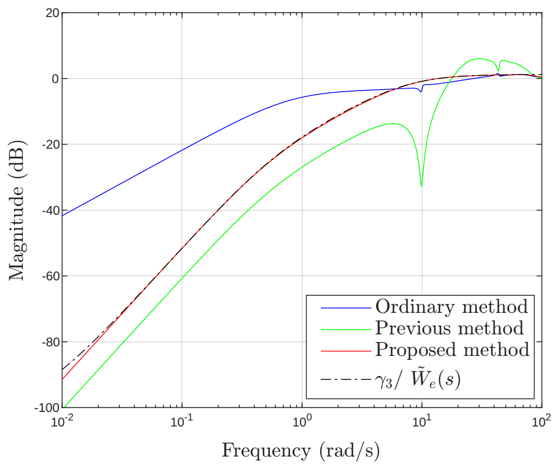


Fig. 11. Magnitude plots of nominal transfer functions from d_o to y_3.

In order to confirm the effectiveness of the sensitivity shaping by the proposed method, the step responses are evaluated when the mass m_2 of Cart 2 varies from 0.2 to 0.22. This corresponds to a situation where the natural frequency $\sqrt{k_2/m_2}$ decreases from about 38.7 [rad/s] to 36.9 [rad/s]. The step responses for the designed controllers are compared to confirm how this variation affects the behavior of the system. The results for cases (i), (ii), and (iii) are shown in Figs. 12, 13, and 14, respectively. In case (i) shown in Fig. 12, designed with constant weights $W_{od}(s)$, there is little difference in response compared to Fig. 7, even with variations in m_2. Figure 13 shows that the controller designed using the integral weight $W_{id}(s)$ becomes unstable when the mass m_2 increases, indicating that it cannot maintain robust stability against changes in the dynamics of the plant. On the other hand, in case (iii) shown in Fig. 14, the step response is stable and does not show variations even when

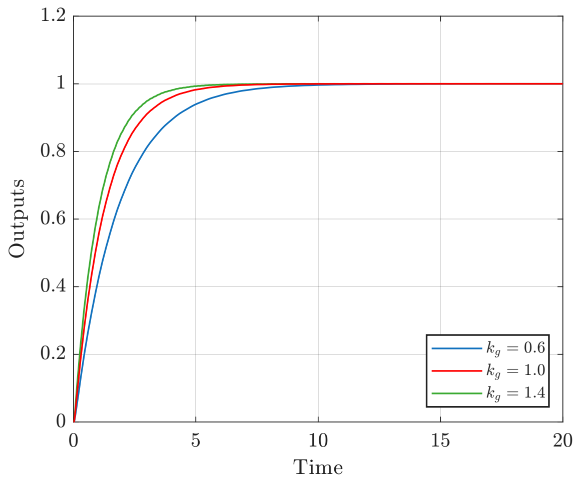


Fig. 12. Step responses achieved by ordinary method.

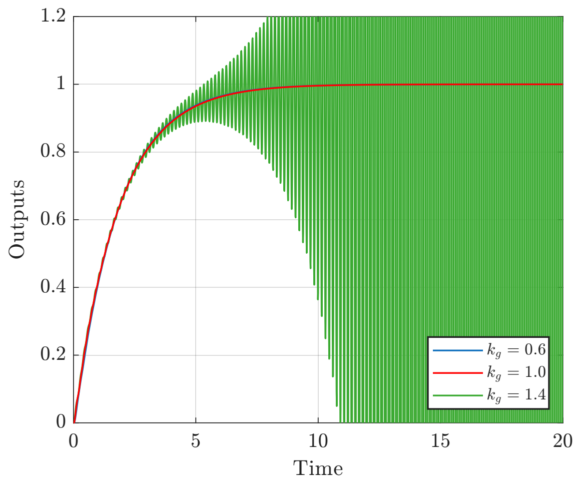


Fig. 13. Step responses achieved by previous design method.

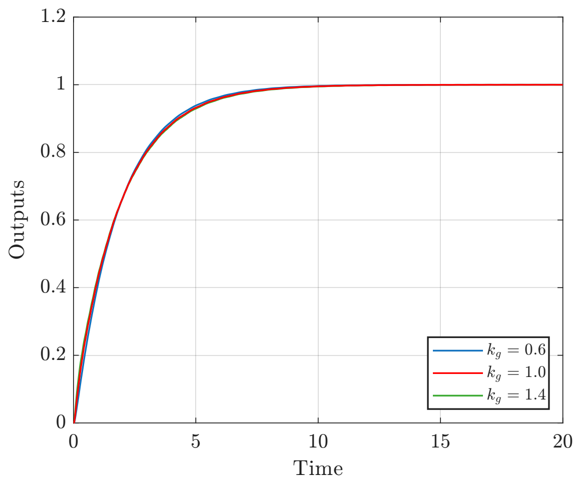


Fig. 14. Step responses achieved by proposed method.

the mass fluctuates. This is because the gain of the sensitivity function is suppressed to a lower value than in the previous method in the neighborhood of the natural frequency of cart 2 after the fluctuation. These results confirm that the proposed method achieves type 2 sensitivity characteristics and suppresses response deterioration caused by sensitivity peaks in the high-frequency range, a problem with previous methods.

5. CONCLUSIONS

This paper proposes an H_∞ controller design method for 2-DOF control systems. The proposed method directly shapes the output sensitivity functions as type 2 in type 1 servo systems. However, if the output sensitivity is excessively reduced in shaping, the tracking performance, such as quick response, and even the stability of systems will deteriorate. Therefore, it is necessary to conduct a detailed analysis to evaluate whether the trade-off between output sensitivity and tracking performance can be designed in a well-balanced design.

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APPENDIX

The designed part of the controller $K(s)$ is given by the following state-space representation.

$$\dot{x}_k = A_k x_k + B_k y_k, \tag{10a}$$

$$u_k = C_k x_k + D_k y_k, \tag{10b}$$

$$y_k = [y_1\, y_2\, y_3]^{\mathrm{T}}, \tag{10c}$$

$$u_k = [u_1,\, u_2]^{\mathrm{T}} \quad (\text{or} = u_1). \tag{10d}$$

The coefficient matrices designed by methods (i), (ii), and (iii) are given below.

(i) Ordinary method.

$$A_k = \begin{bmatrix} 13 & -7.211 & -37.05 & 68.24 \\ -46.77 & -93.19 & -7.994 & 19.4 \\ 94.8 & -15.03 & -134.4 & 182.2 \\ 66.27 & -25.39 & -93.82 & 52.97 \end{bmatrix}, \tag{11a}$$

$$B_k = \begin{bmatrix} -2.238 & -4.258 & -13.8 \\ 166.4 & 10.29 & -289 \\ -124.6 & -8.802 & 203 \\ -98.04 & -6.214 & 134.8 \end{bmatrix}, \tag{11b}$$

$$C_k = \begin{bmatrix} -1.72 & -9.193 & -16.66 \end{bmatrix}, \tag{11c}$$

$$D_k = \begin{bmatrix} -16.33 & 1.199 & -24.65 \end{bmatrix}. \tag{11d}$$

(ii) Previous design method.

$$A_k = \begin{bmatrix} -1.478 & 19.38 & -54.67 & -13.99 \\ -199.9 & -192.1 & 99.6 & -93.43 \\ 306.7 & 260 & -191.1 & 170.6 \\ 478.3 & 490.7 & -107.9 & 206.9 \end{bmatrix}, \tag{12a}$$

$$B_k = \begin{bmatrix} 25.78 & -0.3789 & 8.921 \\ 1.725 & -2.039 & 317.5 \\ 25.1 & 4.26 & -669.2 \\ -111.3 & 5.323 & -574.2 \end{bmatrix}, \tag{12b}$$

$$C_k = \begin{bmatrix} 25.67 & -17.95 & -14.32 & 13.23 \\ 294.8 & 422.3 & 115.4 & 173.8 \end{bmatrix}, \tag{12c}$$

$$D_k = \begin{bmatrix} -0.3762 & 0.4941 & -59 \\ -89.91 & 5.031 & -338.9 \end{bmatrix}. \tag{12d}$$

(iii) Proposed method.

$$A_k = \begin{bmatrix} -318 & 147.2 & 480 & -7.397 \\ -381.1 & 159.8 & 1184 & -10.23 \\ 45.65 & -28.07 & 63.82 & 4.542 \\ 923.3 & -382.2 & -35.51 & -111.9 \end{bmatrix}, \tag{13a}$$

$$B_k = \begin{bmatrix} 664.6 & -155.54 & 547.9 \\ 1447 & -311.7 & 769.7 \\ 47.11 & -4.257 & -57.6 \\ -756.1 & -219.2 & -1753 \end{bmatrix}, \tag{13b}$$

$$C_k = \begin{bmatrix} 22.33 & -9.218 & -7.361 & -1.5 \\ 907.6 & -401.1 & -886.4 & -32.51 \end{bmatrix}, \tag{13c}$$

$$D_k = \begin{bmatrix} -20.47 & 0.9087 & -36.22 \\ -1496 & 177 & -1663 \end{bmatrix}. \tag{13d}$$