A New Limit Cycle Control Method for Multi-modal and 2-dimensional Piecewise Affine Control Systems via State Feedback

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Abstract: The aim of this study is to develop a new limit cycle control method for multi-modal and 2-dimensional piecewise affine control systems via state feedback. Especially, the study focuses on relaxation of design conditions shown in the authors’ previous work by adding a new free parameter. First, matching conditions such that the closed-loop system is equivalent to the reference system with a desired limit cycle are derived. Then, the analytic solution and its existence conditions are shown by solving the matching conditions. A numerical simulation shows that the new control method can generate a desired limit cycle for a given system.

Keywords: Piecewise Affine Systems, Piecewise Affine Oscillator, Limit Cycle Control, State Feedback

1. INTRODUCTION

In general, a limit cycle is known as one of the most famous and important phenomena in nonlinear systems, and there exist a lot of examples of limit cycles in the real world [1]. Various researches on limit cycles have been done from the viewpoints of nonlinear science and control theory in the past [2-11]. Especially, [9, 10] has derived a design method and design conditions of a feedback controller that generates desired limit cycle trajectories for a given piecewise affine system. In addition, the uniqueness and stability of limit cycles are guaranteed mathematically. However, the design conditions are strict, hence it is hard to apply the proposed method to given systems. The aim of this work is to develop a new limit cycle control method in order to relax the design conditions. The contents of this paper is as follows. First, Section 2 gives a problem formulation on limit cycle control. Next, Section 3 develops a new control method by using an additional free parameter. Finally, a numerical simulation is illustrated to confirm the effectiveness of the proposed method in Section 4.

2. PROBLEM FORMULATION

This section shall formulate a problem on generation of desired limit cycles for piecewise affine control systems by state feedback. Consider the 2-dimensional Euclidean space: $\mathbb{R}^2$, its coordinate: $x = [x_1, x_2]^T \in \mathbb{R}^2$, and the origin of $\mathbb{R}^2$: $O$. Let us set $N$ ($N \geq 3$) points $P_i \neq O$ ($i = 1, \ldots, N$) in $\mathbb{R}^2$ and denote the vector from $O$ to $P_i$ by $p_i = [p_{i1} \ p_{i2}]^T$. We also denote the angle between the half line $OP_i$ and the $x_1$-axis by $\theta_i$. Now, without loss of generality, we assume that the points $P_1, \ldots, P_N$ are located in the counterclockwise rotation from the $x_1$-axis, that is, $0 \leq \theta_1 < \cdots < \theta_N$ holds. Next, we define the semi-infinite region $D_i$ which is sandwiched by the half lines $OP_i$ and $OP_{i+1}$ and the line segment $C_i$ joining $P_i$ and $P_{i+1}$, where $P_{N+1} = P_1$.

\[ \text{Set a polygon as a union of } C_i: \]
\[ C := \bigcup_{i=1}^{N} C_i. \]

Fig. 1 shows an example of a polygonal closed curve for $N = 5$.

We next consider the piecewise affine control system defined in $D_i$:
\[ \dot{x} = a_i + A_i x + b_i u, \quad x \in D_i, \]
where $u \in \mathbb{R}$ is the control input and $b_i \in \mathbb{R}^2$ is the coefficient vector for the control input. We also consider the state feedback law in $D_i$:
\[ u = k_i x + l_i, \quad x \in D_i, \]
where $k_i \in \mathbb{R}^2$ and $l_i \in \mathbb{R}$. Substituting (3) into (2), we obtain the closed-loop system:
\[ \dot{x} = a_i + b_i l_i + (A_i + b_i k_i) x, \quad x \in D_i. \]

For the closed-loop system (4), we deal with the next problem on generating desired limit cycles.
Problem 1: For the closed-loop system (4) that consists of a given piecewise affine control system (2) and a state feedback law (3), design the gains of (3): $k_i$, $l_i$ ($i = 1, \ldots, N$) such that a given polygonal closed curve $C$ is a unique and stable limit cycle of (4).

3. MAIN RESULT

This section will derive a new control method as a solution to Problem 1. The main result of this study is obtained as follows.

Theorem 1: Assume that the $N$-modal and 2-dimensional piecewise affine control system (2) and the polygonal closed curve $C$ (1) satisfy

$$b_i^T \neq 0, \forall i \in \{1, \ldots, N\},$$

$$\det[b_i, p_i - p_{i+1}] \neq 0, \forall i \in \{1, \ldots, N\},$$

$$b_i^T (p_i - p_{i+1}) \neq 0, \forall i \in \{1, \ldots, N\},$$

where

$$\begin{aligned}
\omega_i &= 13, \\
k_i &= 5.0, \\
l_i &= 11, \\
k_i &= [1.41, 6.97], \\
l_i &= 0.424, \\
k_i &= [1.0417, 1.0417], \\
l_i &= 6.75.
\end{aligned}$$

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Moreover, the conditions on existence, uniqueness, and stability of the limit cycle are given by (12)–(17) from [11].

The merits of Theorem 1 in comparison with the previous work [10] are as follows; (i) the existence conditions (5)–(8) are represented in the simple forms and are easy to calculate, (ii) the gains of the state feedback law (18)–(20) are obtained in the explicit forms and are also easy to calculate, (iii) the free parameter $s_i$ ($i = 1, \ldots, N$) can be chosen in order to satisfy the conditions (12)–(17), (iv) the rotational directions and the period of limit cycle solutions of (4) can be calculated by the results shown in [11].

4. SIMULATIONS

In this section, a numerical simulation is performed to verify the proposed method. Consider the case where $N = 5$ and $P_1 = (2, 0)$, $P_2 = (0, 2)$, $P_3 = (-2, 2)$, $P_4 = (-2, -2)$, $P_5 = (1, -1)$. The polygonal closed curve $C$ is depicted in Fig. 2. We also consider a 5-modal piecewise affine control system:

$$D_1: \dot{x} = \begin{bmatrix} 1 & 3 \\ a_1 & b_1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ A_1 & b_1 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} u,$$

$$D_2: \dot{x} = \begin{bmatrix} 0 & 3 \\ a_2 & b_2 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u,$$

$$D_3: \dot{x} = \begin{bmatrix} -4 & 0 \\ a_3 & b_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ A_3 & b_3 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} u,$$

$$D_4: \dot{x} = \begin{bmatrix} 2 & 2 \\ a_4 & b_4 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ A_4 & b_4 \end{bmatrix} x + \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} u,$$

$$D_5: \dot{x} = \begin{bmatrix} 3 & 4 \\ a_5 & b_5 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ A_5 & b_5 \end{bmatrix} x + \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} u,$$

and it turns out that the system (21) satisfies the existence conditions (5)–(8). Then, we have

$$\begin{aligned}
\omega_1 &= 11.75, \quad \lambda_1 = 2.25, \quad \mu_1 = 1, \\
\omega_2 &= 5, \quad \lambda_2 = 1.25, \quad \mu_2 = 2, \\
\omega_3 &= 11.75, \quad \lambda_3 = 1.4375, \quad \mu_3 = 0.5, \\
\omega_4 &= 11.2, \quad \lambda_4 = 0.15, \quad \mu_4 = 0.005, \\
\omega_5 &= 1.3333, \quad \lambda_5 = 0.9374, \quad \mu_5 = 0.0625,
\end{aligned}$$

We can see that (22) satisfy (16), and the gains are calculated as

$$\begin{aligned}
k_1 &= [6 6], \quad l_1 = 6.6667, \\
k_2 &= [1 6], \quad l_2 = -13, \\
k_3 &= [5 0], \quad l_3 = 11, \\
k_4 &= [-1.41 - 6.97], \quad l_4 = 0.424, \\
k_5 &= [-1.0417 - 1.0417], \quad l_5 = -6.75.
\end{aligned}$$
\[
\omega_i = \left\{ \begin{array}{ll}
(I) & : \frac{b_2 a_1 - b_1 a_2^2 - (b_1 p_{2,i+1} - p_{2,i}) (p_{2,i+1} - p_{2,i}) + b_2^2 (p_{1,i+1} - p_{1,i}) (p_{1,i+1} - p_{1,i})}{(p_{1,i+1} - p_{1,i})^2 - (p_{1,i+1} - p_{1,i})^2} \\
(II) & : \frac{1}{p_{1,i+1} - p_{1,i}} \left( \frac{b_1}{b_2} a_1^2 - a_2 \right) - \frac{b_1 p_{2,i+1} - p_2 p_{2,i+1}}{(p_{1,i+1} - p_{1,i})^2} \left( A_{11}^{22} - \frac{b_1}{b_2} \right)
\end{array} \right.
\]

\[
\lambda_i = \left\{ \begin{array}{ll}
(I) & : s_i \\
(II) & : s_i \\
(III) & : s_i
\end{array} \right.
\]

\[
\mu_i = \left\{ \begin{array}{ll}
(I) & : (p_{1,i+1} - p_{1,i}) (p_{2,i+1} - p_{2,i}) s_i - A_{11}^{11} \\
(II) & : (p_{1,i+1} - p_{1,i}) (p_{2,i+1} - p_{2,i}) s_i - A_{12}^{12} \\
(III) & : (p_{1,i+1} - p_{1,i}) (p_{2,i+1} - p_{2,i}) s_i - A_{11}^{11}
\end{array} \right.
\]

From Theorem 1, the polygonal closed curve $C$ is guaranteed as the unique and stable limit cycle of the closed-loop system. The simulation results with the initial state $x_0 = (2, 2)$ as an exterior point of $C$ are depicted in Figs 3 and 4. Fig. 3 illustrates the solution trajectory on the $x_1, x_2$-plane. In Fig. 4, the time histories of $x_1$ and $x_2$ are shown, respectively. From these results, it turns out that the solution trajectory that starts from $x_0$ behaves as a limit cycle for the desired polygonal closed curve $C$. In addition, the rotational direction and the period of the solution trajectory is counterclockwise and $T = 1.21$, respectively, and they agree with the analysis results shown.
5. CONCLUSIONS

This study has developed a new control method for multi-modal and 2-dimensional piecewise affine systems. Especially, the design conditions have been relaxed by using an additional free parameter in comparison with the authors’ previous work.

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