Analysis of the Inverse Kinematics in Roof-type Control Moment Gyros

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Abstract: The system of roof-type control moment gyros (CMGs) has been extensively investigated in the field of spacecraft attitude control. Although the system has been implemented in actual satellites, singularity is still a subject of academic study. This paper analyses the inverse kinematics of the roof-type CMGs system. Three approaches for computing all solutions of inverse kinematics are presented, and then, the similarity and the difference of those approaches are discussed.

Keywords: spacecraft; attitude control; control moment gyro; inverse kinematics

1. INTRODUCTION

Singularity of the system of Control Moment Gyros (CMGs) has been extensively studied in the field of spacecraft attitude control [1]. A kind of regularization of Jacobi matrix of angular velocity is a typical approach for overcoming singularity issues. The authors have proposed an alternative approach called the Inverse Kinematics Steering Logic (IKSL) for the pyramid-type CMGs system [2]. The main idea of IKLS lies in the Newton method on the freedom of the inverse kinematics from the angular momentum to the gimbal angles [2, 3]. Because the Newton method assumes some sort of smoothness, a natural question arises on the topological structure of the solutions of inverse kinematics.

To investigate the topological structure, the authors have firstly studied the computation method of inverse kinematics for the two CMG system with a common gimbal axis and then applied to the visualization method of the solutions of the inverse kinematics for the three CMGs system, which consists of three CMGs with a common gimbal axis [4]. The solutions consist of two connected curves inside the 1H singular surface, consist of one connected curve outside the 3H singular surface, and shrink to the one point on the 3H singular surface.

In this paper, we investigate the topological structure of the roof-type CMGs system. The roof-type CMGs system is a special case of the pyramid-type CMGs system but is important because the roof-type CMGs system has been implemented in actual satellites such as WorldView-2. The computation method in [2] for the pyramid-type CMGs system can be applicable to the roof-type CMGs system; specifically in two ways: one is direct variable elimination from the roof-type CMGs system and the other is indirect variable elimination by restricting the skew angle of the pyramid-type CMGs system to be 90 degree. The roof-type CMGs system can be recognized as the combination of two CMGs systems. The computation method in [4] for the two CMGs system can be applicable to the roof-type CMGs system. We compare the similarity and difference of those computational procedures. The preliminary version of this paper has been presented in [5].

2. ROOF-TYPE CMGS SYSTEM

Consider the roof-type CMGs system as shown in Fig. 1. Let CMG-1 denote the i-th CMG, (i = 1, 2, 3, 4). The pair of CMG-1 and CMG-2 is denoted by CMG-1,2 and the pair of CMG-3,4 is denoted by CMG-3,4. The axis of CMG is fixed in the body fixed frame and the gimbal axis of CMG-1 is denoted by the unit vector g1 as shown in Fig. 1.

The gimbal angle of CMG-1 is denoted by θ1, and the gimbal angles are collectively written as \( \theta = [\theta_1, \theta_2, \theta_3, \theta_4]^T \), where the superscript T is the transpose of the matrix. The angular momentum of the wheel of CMG-i is denoted by \( h_i \):

\[
\begin{align*}
  h_1 &= \begin{bmatrix} \cos \theta_1 & 0 & -\sin \theta_1 \end{bmatrix}^T, \\
  h_2 &= \begin{bmatrix} \cos \theta_2 & 0 & -\sin \theta_2 \end{bmatrix}^T, \\
  h_3 &= \begin{bmatrix} \cos \theta_3 & \sin \theta_3 & 0 \end{bmatrix}^T, \\
  h_4 &= \begin{bmatrix} \cos \theta_4 & \sin \theta_4 & 0 \end{bmatrix}^T,
\end{align*}
\]

where the magnitudes of \( h_i \) are supposed to be same and normalized be 1. The total angular momentum, \( h_t \), is the sum of \( h_i \):

\[
h_t = h_1 + h_2 + h_3 + h_4 \tag{5}
\]

3. INVERSE KINEMATICS

3.1. Problem formulation

The relation from \( \theta \) to \( h_t \) in Eq. (5) is the kinematics in the roof-type CMGs system. It is obvious that \( h_t \) is uniquely determined from \( \theta \). In contrast, the relation from \( h_t \) to \( \theta \) in Eq. (5) is the inverse kinematics in the roof-type CMGs system. Because of redundancy, \( \theta \) is not uniquely determined from \( h_t \).

Our problem is to find all solutions of the inverse kinematics, i.e., to parametrize \( \theta \) satisfying Eq. (5) for given \( h_t \). Three computational procedures are presented and compared in the subsequent of this paper.
3.2. Direct variable elimination method

Let us discuss the computation method along [2]. To eliminate the redundancy, \( \theta_4 \) in \( \Theta \) is temporarily fixed. Then the number of equations and variables are balanced. We can directly eliminate the variable \( \theta_4 \) from the equations of total angular momentum in Eq. (5).

Let the left-hand side of Eq. (5) be denoted by \( h_1 - h_4 =: [X, Y, Z]^T \) for notational simplicity. \( \sin \theta_i \) and \( \cos \theta_i \) in the right-hand side of Eq. (5) are replaced by new variables \( \sin \theta_i =: s_i \), \( \cos \theta_i =: c_i \). Equation (5) is written as

\[
-c_1 - c_2 - c_3 + X = 0, \tag{6}
\]

\[
-s_3 + Y = 0, \tag{7}
\]

\[
s_1 + s_2 + Z = 0. \tag{8}
\]

The Pythagoras theorem is written as

\[
s_1^2 + c_1^2 - 1 = 0, \tag{9}
\]

\[
s_2^2 + c_2^2 - 1 = 0, \tag{10}
\]

\[
s_3^2 + c_3^2 - 1 = 0. \tag{11}
\]

The number of equations and variables are balanced again.

Equations (6)–(11) are multivariable polynomial equations. They can be simplified based on the theory of Groebner basis. By eliminating variables, \( c_1, s_3, c_2, s_2, c_1 \), we have the fourth order polynomial of \( s_1 \) as follows:

\[
M_4 s_1^4 + M_3 s_1^3 + M_2 s_1^2 + M_1 s_1 + M_0 = 0, \tag{12}
\]

where the coefficients \( M_i, (i = 0, \ldots, 4) \) are the constants depending on the given \( h_i \) and the temporality fixed \( \theta_4 \). Once the solutions of Eq. (12) are obtained the other variables, \( c_3, s_3, c_2, s_2, c_1 \), can be computed in turn by using Eqs. (6)–(11). \( \theta_1, \theta_2, \theta_3 \) are obtained by computing the four quadrant inverse tangent for \( s_1, s_2, c_2, s_3, c_3 \).

All solutions of inverse kinematics can be obtained by changing the temporarily fixed \( \theta_4 \) on \( -\pi < \theta_4 \leq \pi \).

3.3. Indirect variable elimination method

Let us also discuss the computation method along [2]. If we apply the computational method for the pyramid-type CMGs system in [2] for the roof-type CMGs system, we can indirectly eliminate the variable \( \theta_4 \) from the equations of total angular momentum. Then, the eighth order polynomial of \( s_1 \) is obtained as follows:

\[
\begin{align*}
N_8 s_1^8 + N_7 s_1^7 + N_6 s_1^6 + N_5 s_1^5 + N_4 s_1^4 + N_3 s_1^3 + N_2 s_1^2 + N_1 s_1 + N_0 &= 0, \tag{13}
\end{align*}
\]

where the coefficients \( N_i, (i = 0, \ldots, 8) \) are the constants depends on the given \( h_i \) and the temporality fixed \( \theta_4 \). The rest of the procedure is analogous to the direct variable elimination in the previous subsection.

3.4. Distribution method

Let us note that the roof-type CMGs system can be recognized as the combination of two CMGs systems. The second component of \( h_i \) consists of CMG-3,4 and the third component consists of CMG-1,2. Therefore, if the first component is distributed to CMG-1,2 and CMG-3,4, the computation of inverse kinematics for the roof-type CMGs system is decomposed to the computations of inverse kinematics for CMG-1,2 and CMG-3,4.

All solutions of inverse kinematics can be obtained by changing the ratio of distribution in the first component of \( h \) to CMG-1,2 and CMG-3,4.

3.5. Comparison

Let us compare the direct variable elimination method and indirect elimination method. Although the order of polynomials in Eq. (12) and Eq. (13) are formally different, those equations indeed agree. In particular, it is shown that the coefficients, \( N_5, N_6, N_7, \) and \( N_8, \) in Eq. (13) are surprisingly equal to zero when the skew angle of 90 degree is substituted into the pyramid-type CMGs system.

Let us compare the direct variable elimination method and distribution method. Comparison by analytic calculation is difficult to pursue, and therefore, comparisons by numerical computation are carried out for several examples. It is observed that the results of numerical computations coincide for most of the cases, e.g., \( h_i = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}^T \). However, it is surprising that Eq. (12) is not defined for several cases, e.g., \( h_i = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \) and \( h_i = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T \), because all coefficients \( M_i \) in Eq. (12) becomes zero for such cases. The direct variable elimination method therefore cannot be applicable for such cases. In contrast, the distribution method can be applicable for any cases. Further numerical examples will be presented in the conference presentation.

4. CONCLUSION

Three computational procedures are presented for computing all solutions of inverse kinematics for the roof-type CMGs system. The distribution method is the best approach among three procedures because the direct elimination method and indirect elimination method cannot be applied for certain cases. Further analysis and ap-
plication to the steering logic are expected in future re-
search.

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